WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET I (2025–2026)

1. Find the least positive integer N such that the numbers $1, 2, \ldots, N$ can be divided in 2025 groups with the sum of numbers in each group being the same. Each group must have at least one number in it, each number must be in exactly one group.

SOLUTION. The least value of N is $N = 2 \times 2025 - 1 = 4049$. First, this value works because we have

$$4049 = 4048 + 1 = 4047 + 2 = \cdots = 2025 + 2024,$$

which gives 2025 groups: one single-number group 4049 and 2024 groups k, $4049 - k$ for k between 1 and 2024.

Now let us see that no lessor value of N can work. Indeed, there can be no more than a single one-number group, because two groups consisting of one number each will not have the same sum. The remaining groups must therefore have at least two numbers each, giving the minimal possible value of $1 + 2 \times 2024 = 4049$.

2. Is it possible to paint all the integers that are at least 2 using three colors: yellow, green, and purple, in such a way that each number is painted in a single color, not all numbers are painted the same, and whenever two numbers are of different colors, their product is painted using the third color? (For example, if 3 is yellow and 8 is purple, $3 \cdot 8 = 24$ must be green.)

SOLUTION. It is not possible. The basic idea is that if we keep following this coloring rule, we will eventually reach a number which can be factored in two different ways, giving two different choices of color for that number. We give two possible ways to see this.

First approach. Our first observation is that for the above coloring to be possible, a number and its square have to be assigned the same color. To see this, let's argue by contradiction. Suppose $n > 1$ was a positive integer assigned color yellow (Y), and n^2 was assigned color green (G). Then by our coloring rule

$$\begin{aligned}
n^3 &= n \times n^2 = \text{Y} \times \text{G} = \text{PURPLE (P)} \\
n^4 &= n \times n^3 = \text{Y} \times \text{P} = \text{G} \\
n^5 &= n \times n^4 = \text{Y} \times \text{G} = \text{P} \\
&= n^2 \times n^3 = \text{G} \times \text{P} = \text{Y},
\end{aligned}$$

and we have reached a contradiction since n^5 cannot be assigned both PURPLE and YELLOW.

Next, we choose two numbers m and n assigned two different colors, say Y and G, respectively. By our previous observation, m^2 and n^2 are also assigned the colors Y and G, respectively. Once more, by repeated use of our coloring rule, we get

$$\begin{aligned}
mn &= m \times n = \text{Y} \times \text{G} = \text{P} \\
mn^2 &= m \times n^2 = \text{Y} \times \text{G} = \text{P} \\
&= mn \times n = \text{P} \times \text{G} = \text{Y}.
\end{aligned}$$

This is again a contradiction since mn^2 cannot be assigned both PURPLE and YELLOW. We thus reach the conclusion that a coloring with the stipulated rules is not possible.

Second approach. Let us choose two numbers of different colors. Say, n is PURPLE and m is GREEN. We can now deduce the colors of various other numbers:

$$nm = n \times m = PURPLE \times GREEN$$

must be YELLOW, which implies that

$$nm^2 = nm \times m = YELLOW \times GREEN$$

is PURPLE, and

$$nm^3 = nm^2 \times m = PURPLE \times GREEN$$

is YELLOW. Now looking at two products

$$nm^2 = n \times m^2$$

and

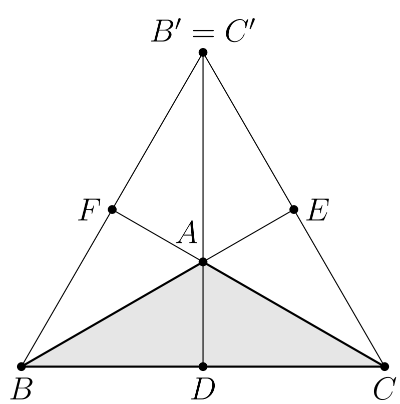
$$nm^3 = nm \times m^2,$$

we see that there is no legal choice for the color of m^2: the only way to satisfy the first identity is for m^2 to be PURPLE, while the only way to satisfy the second is for it to be YELLOW.

3. In $\triangle ABC$ let B' be the reflection of vertex B across line AC, and let C' be the reflection of vertex C across line AB. Suppose that $B' = C'$. Find the angle $\angle BAC$.

SOLUTION. There are many ways to solve this problem. First, because C' is the reflection of C in line AB, it follows that $\angle BAC \cong \angle BAC'$. Similarly, $\angle CAB \cong \angle CAC' = \angle CAB'$. Thus, $\angle BAC = \angle CAB' = \angle BAB'$, and since the sum of these is equal to $360°$, each of these is $\frac{1}{3} \cdot 360° = 120°$.

Alternatively, there are many ways to see that $\triangle BCB'$ must be equilateral. Let D, E, and F be the midpoints of $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively, as shown. Because C' is the reflection of C across line AB, it follows that $BC = BC' = BB'$. Similarly, $BC = CB'$, implying that the three sides of $\triangle BCB'$ have the same length.



Here is another way to see that $\triangle BCB'$ is equilateral. Because C' is the reflection of C across line AB, it follows that $\triangle BCE \cong \triangle BC'E$, so $\angle BC'C = \angle BCC'$. Similarly, $\angle CB'B = \angle CBB'$. Thus, the three angles in $\triangle BCB'$ are equal, implying that $\triangle BCB'$ is equilateral.

Once we know that $\triangle BCB'$ is equilateral, there are many ways to show that $\angle BAC = 120°$. Because $BF = B'F$ and $CE = C'E$, it follows that E and F are midpoints of $\overline{CB'}$ and $\overline{BC'}$, respectively, so $\overline{BE}$ and $\overline{CF}$ are medians of $\triangle BCB'$, implying that A is the centroid of $\triangle BCB'$.

Because $\triangle BCB'$ is equilateral, $\angle BAC = \angle CAB' = \angle BAB'$, as in the first solution. Alternatively, $\overline{BE}$ and $\overline{CF}$ are angle bisectors of $\angle CBB'$ and $\angle BCB'$, respectively, implying that $\angle CBA = \angle BCA = \frac{1}{2} \cdot 60° = 30°$. Thus, $\angle BAC = 180° - 2 \cdot 30° = 120°$.

4. Consider this 4×4 grid filled with integers. We want to go from the upper left corner square to the lower right corner square in a way that we never visit a square more than once, and we always step to a neighboring square (up, down, left or right). We add up the numbers that we see along our path. What is the greatest sum that we can achieve?

5	12	3	16
9	1	14	6
11	8	2	13
10	4	15	7

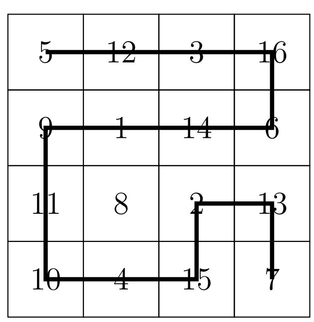
SOLUTION. We show that the greatest sum we can collect is 128.

Color the squares of the grid in a chessboard pattern:

5	12	3	16
9	1	14	6
11	8	2	13
10	4	15	7

Note that we start from a white square and end up at a white square. Moreover, in each step the color of the square we are on changes: we go from gray to white and white to gray. Because of this, if we start and end on a white square, then we visited exactly one fewer gray squares than white squares, which means that we did not visit at least one gray square. The least number on a gray square is the number 8, so the total sum that we collected cannot be more than the sum of all numbers minus 8: $1 + 2 + \cdots + 16 - 8 = 128$.

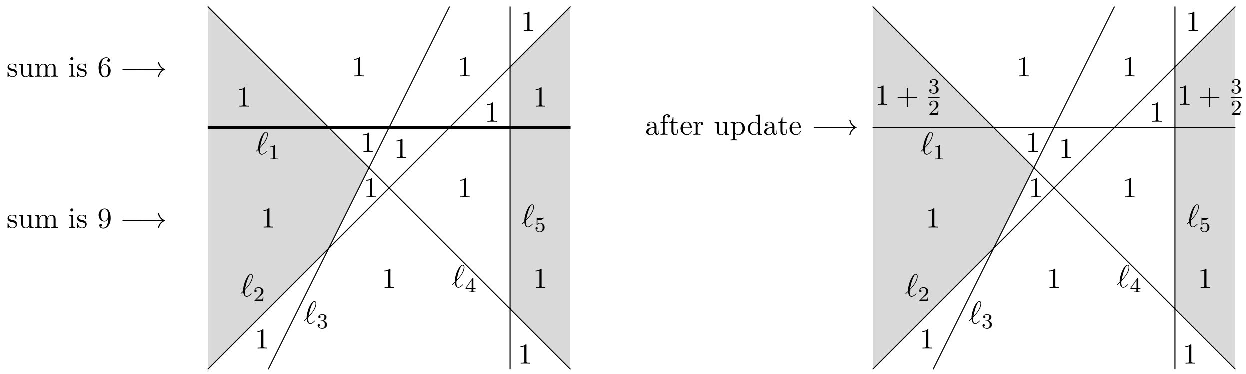
But this sum can be achieved as the path below shows.



5. Several lines are drawn in a plane so that none of the lines are parallel, and there are no three lines going through the same point. The lines divide the plane into several domains. Show that one can put positive numbers in each of those domains so that for each line the the sums of numbers are the same on the two sides.

SOLUTION. Start by putting a positive number, say 1, in each domain. After that, adjust the numbers to satisfy the condition in the following way.

Let us enumerate the lines from 1 to n: $l_1, l_2, ..., l_n$. Notice that for each line there are four unbounded domains, two on each side of the line, such that the boundary of the domain contains a ray from that line. Starting with l_1, calculate the total sums on each side of the line. (See the illustration below.) Increase the numbers in the two unbounded domains on the side where the sum is less by equal amounts, so that the new sums on the two sides of the line are equal. After that, do the same for l_2, l_3, and so on.



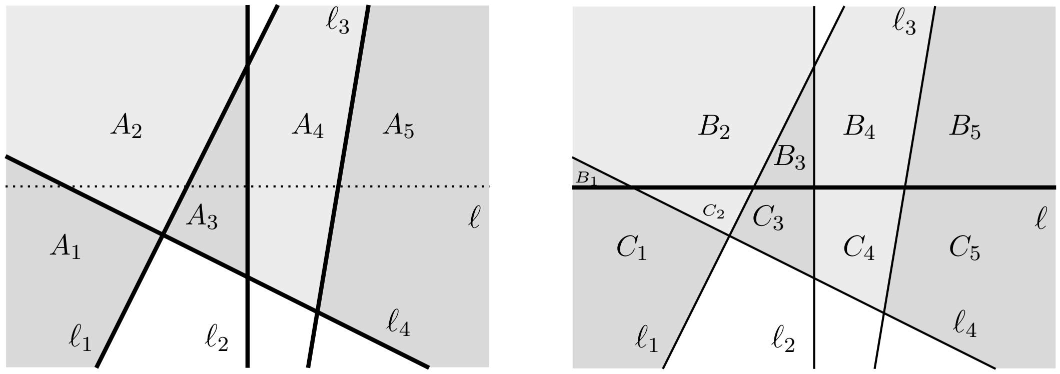
Notice that on the k-th step, the two unbounded regions where the numbers are increased are situated on the opposite sides of each line, except the line l_k. Indeed, since no lines are parallel, every line l_m, $m \neq k$ intersects the k-th line. Since l_m cannot intersect interiors of the two unbounded domains, l_m must intersect l_k at a point from the segment of l_k which lies between the two domains (in the case $n = 2$ the segment reduces to a point). Since, once again, l_m does not intersect the interiors of the domains, they must lie on the opposite sides of l_m.

Hence, after each step the sums that were fixed to be equal on previous steps will remain equal. Therefore, after n steps we will achieve the desired property.

Similar solution with induction: We will prove the statement with induction on the number of lines. When we have a single line drawn, then we just have two domains (one on each side of the line). In this case we can write the same number (e.g. 1) in both domains to satisfy the requirements.

Assume that the statement is true for any n lines drawn in the plane, we will show that it has to be true for $n+1$ lines as well. Suppose that we have $n+1$ lines drawn (with none parallel, and no three lines going through the same point), let one of them be called ℓ, denote the others by $\ell_1, \dots, \ell_n$. If we remove ℓ from the picture, we can use our induction hypothesis for the remaining n lines which means that we can put positive numbers in the created domains to satisfy the requirements.

Now draw ℓ back! It intersects each of the other n lines (since there are no parallel lines, or more than 2 lines going through a point). This means that ℓ will go through exactly n intersection points (one with each of the other n lines). These n points divide ℓ into $n+1$ parts which we denote by $I_1, \dots, I_{n+1}$: two semi-infinite intervals ('rays') at the two ends (I_1 and I_{n+1}), and $n-1$ finite intervals (in the middle, $I_2, \dots, I_n$). Each one of these $n+1$ intervals will divide a domain (coming from the first n lines) in half. Denote these $n+1$ domains in order (from one end of ℓ to the other) by $A_1, A_2, \dots, A_{n+1}$. Each will be divided into two new domains by ℓ. Denote the domains created from A_i by B_i and C_i, in a way that $B_1, \dots, B_{n+1}$ are on the same side of ℓ. (See the picture below.)



Denote the number that we had in domain A_i by a_i. When we draw the line ℓ, erase the numbers from the domains $A_1, \ldots, A_{n+1}$, and for each $1 \le i \le n+1$ put $\frac{1}{2}a_i$ into the newly created domains B_i and C_i. In other words: if ℓ divides a domain into two, then we just distribute the number that we had in that domain within the created two new domains (equally). This way it is still true that for each $1 \le j \le n$ the sum of numbers on the two sides of ℓ_j are the same. Indeed: when we draw the line ℓ, the sums on the two sides of ℓ_j will not change: if we divide a domain into two by ℓ, then the sum of numbers in the newly created domains will be the same as the one we had before.

However, it is not clear that the sums are the same on the two sides of ℓ. We will show that we can modify the numbers in a way so that we do not ruin the balance for $\ell_1, \ldots, \ell_n$, and make the sums on the two sides of ℓ the same. Compute the sums for the two sides of ℓ. If they are the same, we are done; we found an appropriate collection of numbers. If they are not the same, then one side (e.g. the one containing the newly created domains $B_1, \ldots, B_{n+1}$) is greater by some positive number x. Now increase the numbers in C_1 and C_{n+1} by $x/2$ each. This way the sum on both sides of ℓ are the same. We will show that even with these changes, the two sides of ℓ_j have the same sums, for each $1 \le j \le n$. To show this, we just need to show that C_1 and C_{n+1} are on two different sides of ℓ_j: because we increased both by the same amount $(x/2)$, the sums will remain the same. This statement is a consequence of the following observation: suppose that we color one side of ℓ_j with white, the other with black. The line ℓ intersects ℓ_j, so half of it (a semi-infinite interval) will be white, and the other half of it will be black. In particular, the two semi-infinite intervals I_1 and I_{n+1} at the two ends of ℓ must be of different colors. But this means that A_1 and A_{n+1} were on two different sides of ℓ_j, which also implies that C_1 and C_{n+1} are on different side as well, completing the proof.