WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET III (2024–2025)

1. Four nonzero real numbers a, b, c, d satisfy the equation

$$\frac{a}{b} + \frac{b}{a} = \frac{c}{d} + \frac{d}{c}.$$

Show that the product of two of the numbers from a, b, c, d is equal to the product of the other two.

SOLUTION 1. Let us rewrite the condition in the form

$$\frac{a}{b} + \frac{b}{a} = \frac{c}{d} + \frac{d}{c} \Leftrightarrow \frac{a}{b} - \frac{c}{d} = \frac{d}{c} - \frac{b}{a} \Leftrightarrow \frac{ad - bc}{bd} = \frac{ad - bc}{ac}.$$

If the numerator of these two fractions are zero, then $ad = bc$. Otherwise, the two fractions have equal numerators, so their denominators are also equal, and this means that $bd = ac$.

SOLUTION 2. Let $x_1 = \frac{a}{b}$ and $x_2 = \frac{c}{d}$. If $x_1 = x_2$, then our equality holds and $ad = bc$. So hereafter assume that $x_1 \neq x_2$. If $x_1 = \frac{1}{x_2}$, then our equality holds and $ac = bd$.

If $x_1 = 1$, then $x_2 + \frac{1}{x_2} = 2$. This leads to the equation $x_2^2 + 1 = 2x_2$ and $x_2^2 - 2x_2 + 1 = (x_2 - 1)^2 = 0$, showing that $x_2 = 1$ and $x_1 = x_2$. Hence, we may assume $x_1 \neq 1$.

Let us denote the value of the left and right parts of the equality by t. Then the condition is rewritten in the form

$$x_1 + \frac{1}{x_1} = x_2 + \frac{1}{x_2} = t.$$

Consider the function $f(x) = x + \frac{1}{x}$. The numbers x_1 and x_2 are both roots of the equation $f(x) = t$. Note that if x is a root, then $1/x$ is also a root.

Let's rewrite this equation in the form

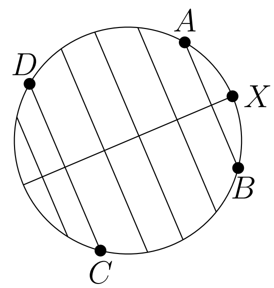
$$f(x) = t \Leftrightarrow x + \frac{1}{x} = t \Leftrightarrow x^2 - tx + 1 = 0.$$

This is a quadratic equation, so it has no more then 2 roots. If x_1 is a root, then $1/x_1$ is also a root, and because $x_1 \neq 1$, we have $x_1 \neq 1/x_1$. Hence x_1 and $1/x_1$ are the only roots. But $x_2 \neq x_1$ is also a root, so we must have $x_2 = \frac{1}{x_1}$.

2. Given a regular 2025-gon, how many ways can we choose four of its 2025 vertices so that they are the four vertices of a trapezoid?

SOLUTION. Let X be any vertex of the regular 2025-gon, and ℓ be the line through X and the center of the 2025-gon. Then ℓ is an axis of symmetry of the 2025-gon. That is, each vertex $A \neq X$ of the 2025-gon can be paired with exactly one other vertex B such that $\overline{AB} \perp \ell$. Because there are 2024 vertices of the 2025-gon that are not X, there must be $\frac{2024}{2} = 1012$ pairs of vertices A and B such that $\overline{AB} \perp \ell$. Any 2 such pairs will give 4 vertices A, B, C, and D, where $\overline{AB} \parallel \overline{CD}$, so those 4 vertices are the vertices of a trapezoid.

Conversely, suppose that A, B, C, and D are vertices of the 2025-gon such that $ABCD$ is a trapezoid with $\overline{AB} \parallel \overline{CD}$. Let ℓ be the perpendicular bisector of $\overline{AB}$. Then ℓ is an axis of symmetry of the 2025-gon and is also a perpendicular bisector of $\overline{CD}$. As a result, ℓ passes through exactly one vertex X of the 2025-gon.



It follows that each trapezoid whose vertieces are vertices of the 2025-gon corresponds to one vertex X, its corresponding axis of symmetry ℓ, and any two distinct pair of vertices $A-B$ and $C-D$, where $\overline{AB}$ and $\overline{CD}$ are both perpendicular to ℓ. There are 2025 choices for X and ℓ. For each X, there are 1012 pairs of vertices symmetric with respect to ℓ. Each choice of X and choice of two pairs of the 1012 pairs of vertices gives the vertices of a unique trapezoid. Thus, there are

$$2025 \cdot \binom{1012}{2} = 2025 \cdot \frac{1012 \cdot 1011}{2} = 1{,}035{,}921{,}150$$

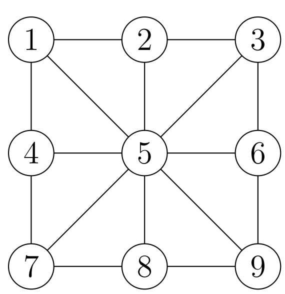
trapezoids whose vertices are vertices of the 2025-gon.

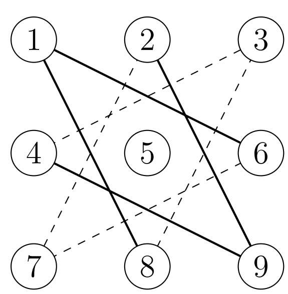
3. A small town has several bus lines. Each line has 3 stops and any two lines share no more than 1 stop. What is the maximum possible number of bus lines if there are 9 bus stops?

SOLUTION. We first show that there can be a maximum of 12 bus lines. Suppose that for some positive integer k there are k bus lines that all share the same bus stop. Because no two bus lines can share more than one stop, each of the k lines has 2 more stops that the other $k-1$ bus lines do not share. This means that these k lines are using $2k+1$ different bus stops. There are only 9 bus stops, so $2k+1 \leq 9$ and $k \leq 4$. Hence, each bus stop can be shared by at most 4 bus lines. Suppose that we have n bus lines, and we write down the bus stops for each line. This gives a list of $3n$ bus stops, and each of the 9 bus stops can appear at most 4 times on the list. Hence, $3n \leq 4 \cdot 9$ giving $n \leq 12$.

To show that 12 bus lines can be achieved, we just have to provide an example. Here is one: number the bus stops with 1, 2, 3, 4, 5, 6, 7, 8, 9. Then the twelve bus lines could use stops 123, 456, 789, 147, 256, 369, 159, 357, 294, 276, 618, 438 which satisfy the conditions of the problem. The following diagram shows how these stops are connected by bus routes. Arranging the 9 bus stops in a 3×3 grid the first 8 lines correspond to the rows, columns and two diagonals. We can check that there are no two lines sharing more than one stops. The remaining 4 are shown in the second diagram: in each line we have exactly one stop from each row and column. Again, it can be checked that no two lines have more than one common stops.

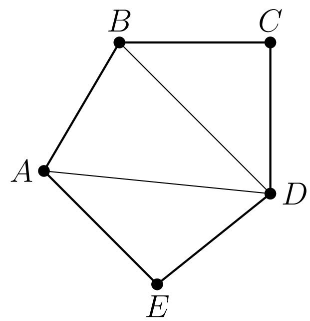
Finally, out of these 4 lines neither one uses more than one stop from a diagonal, row, or column.





4. The convex pentagon $ABCDE$ has equal length sides, but all of its angles are different. Show that the largest and the smallest of its angles share the same side of the pentagon.

SOLUTION. Suppose that the statement is false. Then the smallest and the largest angles have one vertex between them. We can assume that the largest angle is $\angle A$ and the smallest is $\angle C$. Consider the isosceles triangles $\triangle ADE$ and $\triangle BCD$. Notice that side $\overline{AD}$ in $\triangle ADE$ lies opposite to a larger angle $\angle E$ than the side $\overline{BD}$ in $\triangle BCD$, which is opposite to the smallest angle $\angle C$. Therefore $AD > BD$. It follows that in $\triangle BDA$ the angle $\angle ABD$ is greater than the angle $\angle BAD$ because the former lies opposite to a larger side.



By considering the isosceles triangle $\triangle BCD$ we see that

$$\angle CBD = \frac{180° - \angle C}{2}.$$

Similarly,

$$\angle DAE = \frac{180° - \angle E}{2}.$$

Since $\angle C < \angle E$, we conclude that $\angle CBD > \angle DAE$. All in all, we get that

$$\angle B = \angle CBD + \angle ABD > \angle DAE + \angle BAD = \angle A,$$

which contradicts to the assumption that $\angle A$ is the largest angle.

5. Show that the following expression is less than 3:

$$\sqrt{1 + \sqrt{2 + \sqrt{2^2 + \sqrt{2^3 + \sqrt{\cdots \sqrt{2^{2024} + \sqrt{2^{2025}}}}}}}}.$$

SOLUTION 1. Let a_n denote the value of the similar expression with 2^n at the end:

$$\sqrt{1 + \sqrt{2 + \sqrt{2^2 + \sqrt{2^3 + \sqrt{\cdots \sqrt{2^{n-1} + \sqrt{2^n}}}}}}}$$

We will show that the a_n is always less than 2 (then a_{2025} is of course less than 3 as well).

We start by showing that for $n \geq 1$ we have

$$a_{n+1} < \sqrt{1 + \sqrt{2} \cdot a_n}.$$

First note that for $a, b, c > 0$ we have

$$c\sqrt{a+\sqrt{b}} = \sqrt{c^2a + c^2\sqrt{b}} = \sqrt{c^2a + \sqrt{c^4b}}.$$

Iterating this identity we get

$$\sqrt{2}a_n = \sqrt{2}\sqrt{1+\sqrt{2+\sqrt{2^2+\sqrt{2^3+\sqrt{\cdots\sqrt{2^{n-1}+\sqrt{2^n}}}}}}}$$

$$= \sqrt{2+2\sqrt{2+\sqrt{2^2+\sqrt{2^3+\sqrt{\cdots\sqrt{2^{n-1}+\sqrt{2^n}}}}}}}$$

$$= \sqrt{2+\sqrt{2\cdot 2^2+2^2\cdot\sqrt{2^2+\sqrt{2^3+\sqrt{\cdots\sqrt{2^{n-1}+\sqrt{2^n}}}}}}}$$

$$= \sqrt{2+\sqrt{2\cdot 2^2+\sqrt{2^2\cdot 2^4+\sqrt{\cdots\sqrt{2^n\cdot 2^{2^n}+\sqrt{2^{n+1}\cdot 2^{2^{n+1}}}}}}}}$$

$$> \sqrt{2+\sqrt{2^2+\sqrt{2^3+\sqrt{\cdots\sqrt{2^n+\sqrt{2^{n+1}}}}}}}.$$

(Each time we 'move' the original $\sqrt{2}$ multiplier inside a square root we have to square it, so each power of 2 will eventually get multiplied by a number that is at least 2.) Adding 1 to both sides and taking square roots leads to the inequality $\sqrt{1+\sqrt{2}\cdot a_n} > a_{n+1}$.

We have $a_1 = \sqrt{1+\sqrt{2}} < \sqrt{1+3} = 2$. From this our statement now follows from induction, since if $a_n < 2$, then

$$a_{n+1} < \sqrt{1+\sqrt{2}a_n} < \sqrt{1+\sqrt{2}\cdot 2} < \sqrt{1+3} = 2$$

holds as well.

SOLUTION 2. For $1 \le k \le 2024$ denote by b_k the following expression:

$$\sqrt{2^k+\sqrt{2^{k+1}+\sqrt{2^{k+2}+\sqrt{2^{k+3}+\sqrt{\cdots\sqrt{2^{2024}+\sqrt{2^{2025}}}}}}}}$$

Note that we have $b_k = \sqrt{2^k + b_{k+1}}$.

We will show by (reverse) induction that for $2 \le k \le 2024$ we have

$$b_k \le 2^{\frac{k+1}{2}}.$$

For $k = 2024$ this is true because

$$b_{2024} = \sqrt{2^{2024} + \sqrt{2^{2025}}} = \sqrt{2^{2024} + 2^{1012.5}} \leq \sqrt{2 \cdot 2^{2024}} = 2^{\frac{2025}{2}}.$$

For the induction step let $2 \leq k \leq 2023$ and assume that $b_{k+1} \leq 2^{\frac{k+2}{2}}$. Note that if $k \geq 2$, then $\frac{k+2}{2} = \frac{k}{2} + 1 \leq k$, so by our assumption $b_{k+1} \leq 2^k$. Then

$$b_k = \sqrt{2^k + b_{k+1}} \leq \sqrt{2^k + 2^k} = 2^{\frac{k+1}{2}}$$

proving the induction step. Using the inequality $b_2 \leq 2^{\frac{5}{2}}$ we get

$$\sqrt{1 + \sqrt{2 + \sqrt{2^2 + \sqrt{2^3 + \sqrt{\cdots \sqrt{2^{2024} + \sqrt{2^{2025}}}}}}}} = \sqrt{1 + \sqrt{2 + b_2}}$$

$$\leq \sqrt{1 + \sqrt{2 + 2^{5/2}}} < \sqrt{1 + \sqrt{9}} = 3,$$

proving the required bound. ($2 + 2^{5/2} < 9$ follows from $2^{5/2} = \sqrt{32} < \sqrt{36} = 6$.)