WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET II (2024–2025)

1. We have 1000 white unit cubes. We build a $10 \times 10 \times 10$ cube from these, and then paint some (or all) of the faces of this bigger cube purple. After disassembling the bigger cube we found that there are an odd number of unit cubes with at least one purple face. How many faces of the $10 \times 10 \times 10$ cube were painted purple?

SOLUTION. Consider the assembled $10 \times 10 \times 10$ cube with some of the faces painted purple. Suppose that we one-by-one remove the 'top' layer of small cubes for each painted face. After removing all painted layers the remaining cubes are exactly the ones without any purple on them. During the process of removing these layers the original cube remains a rectangular cuboid (a 'box'). With the removal of each layer one of the side lengths is decreased by one. (E.g., after the removal of the first face we are left with a $10 \times 10 \times 9$ rectangular cuboid.) Each side length can only be decreased by at most two along this process since there are at most two painted faces with a given orientation. That means that after removing all the painted faces, we are left with a cuboid of size $a \times b \times c$ where the numbers a, b, and c can be 8, 9 or 10 (and cannot all be 10). The number of remaining cubes is abc, and this can only be odd if $a = b = c = 9$. This can only be achieved if we remove exactly three faces of the original cube, one from each orientation, which means that three of the faces of the original cube were painted purple (and these faces all meet at one of the vertices).

2. The sum of 7 distinct positive integers is at least 100. Show that we can choose 3 of these integers so that their sum is at least 50.

SOLUTION. Label the seven numbers in decreasing order as $a_1 > a_2 > a_3 > a_4 > a_5 > a_6 > a_7$. We will show that $a_1 + a_2 + a_3$ cannot be less than 50.

Assume that $a_1 + a_2 + a_3 < 50$. Since our numbers are integers, we then have $a_1 + a_2 + a_3 \leq 49$. Because the numbers are distinct, the difference between the neighboring numbers is at least one: $a_i \geq a_{i+1} + 1$ for $i = 1, \ldots, 6$. Hence $a_2 \geq a_3 + 1$ and $a_1 \geq a_3 + 2$, which means that if $a_1 + a_2 + a_3 \leq 49$, then $a_3 \leq 15$. Indeed, if $a_3 \geq 16$ then $a_1 + a_2 + a_3 \geq 16 + 17 + 18 = 51$. But if $a_3 \leq 15$, then $a_4 \leq 14$, $a_5 \leq 13$, $a_6 \leq 12$, and $a_7 \leq 11$, which gives

$$a_4 + a_5 + a_6 + a_7 \leq 50.$$

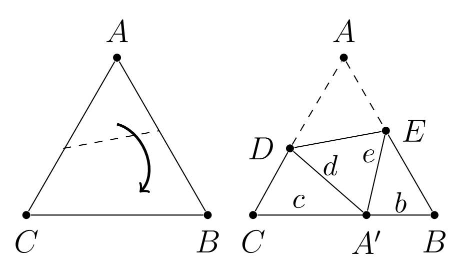
Thus

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 \leq 49 + 50 = 99,$$

which contradicts the assumption of the problem. Hence $a_1 + a_2 + a_3$ must be at least 50.

Remark. Note that our solution shows that we can choose three numbers so that their sum is at least 51. We cannot guarantee a sum of more than 51 as the numbers $12, 13, \ldots, 18$ show.

3. An equilateral triangle $\triangle ABC$ is made out of paper. Fold the paper, as shown on the right, so that vertex A ends up at a point A' on side $\overline{BC}$, and the fold line intersects sides $\overline{AB}$ and $\overline{AC}$ at points D and E, respectively. Let $b = A'B$, $c = A'C$, $d = A'D$, and $e = A'E$. Show that $\dfrac{d}{e} = \dfrac{b + 2c}{2b + c}$.



SOLUTION. Without loss of generality, let the side length of $\triangle ABC$ be 1. Because $\triangle A'DE$ is just the folding of $\triangle ADE$, it follows that $AD = d$ and $AE = e$. Thus, $CD = 1 - d$ and $BE = 1 - e$. Note that $\angle DCA' = \angle A'BE = \angle DA'E = 60°$. Thus, $\angle CA'D$ is the supplement of $\angle BA'D = \angle BA'E + \angle DA'E = \angle BA'E + \angle A'BE$, implying $\angle CA'D$ and $\angle A'EB$ have the same supplements, so they are equal. Because $\triangle A'CD$ and $\triangle EBA'$ have two angles in common, they are similar. From that

$$\frac{A'D}{A'E} = \frac{A'C}{BE} = \frac{CD}{A'B},$$

or equivalently

$$\frac{d}{e} = \frac{c}{1-e} = \frac{1-d}{b}.$$

To finish our proof we will use the algebraic fact that

$$\frac{r}{s} = \frac{x}{y} \quad \text{implies} \quad \frac{r}{s} = \frac{r+x}{s+y},$$

as long as $s + y \neq 0$. This can be seen for example by noting that $\frac{r}{s} = \frac{x}{y}$ implies $xs = yr$, from which we get

$$\frac{r}{s} - \frac{r+x}{s+y} = \frac{r(s+y) - s(r+x)}{s(x+y)} = \frac{yr - xs}{s(x+y)} = 0.$$

Using this implication twice we get

$$\frac{d}{e} = \frac{d+c}{e+(1-e)} = \frac{d+c+(1-d)}{e+(1-e)+b} = \frac{1+c}{1+b}.$$

Substituting $1 = b + c$ gives

$$\frac{d}{e} = \frac{b+2c}{2b+c},$$

as needed.

4. Let $n = 2^{2024} \cdot 3^{2025}$. How many positive divisors does n^2 have that are less than n, but do not divide n?

SOLUTION. If a number m is of the form $2^a \cdot 3^b$ where a and b are positive integers, then the positive divisors of m are exactly the numbers of the form $2^i \cdot 3^j$ with $0 \leq i \leq a$ and $0 \leq b \leq j$. This is because m only has two prime divisors: 2 and 3; hence, any divisor of m must be built from these. This means that $m = 2^a \cdot 3^b$ has $(a+1)(b+1)$ positive divisors, since this is how many different ways we can choose the pair (i, j). Hence, $n = 2^{2024} \cdot 3^{2025}$ has $(2024 + 1) \cdot (2025 + 1)$ positive divisors, and $n^2 = 2^{2 \cdot 2024} \cdot 3^{2 \cdot 2025}$ has $(2 \cdot 2024 + 1) \cdot (2 \cdot 2025 + 1)$ positive divisors.

Note that the positive divisors of n^2 come in pairs: if k divides n^2, then n^2/k also divides it. Unless $k = n$, exactly one of the two numbers k or n^2/k will be less than n. This means that we have

$$\frac{1}{2}\left((2 \cdot 2024 + 1) \cdot (2 \cdot 2025 + 1) - 1\right) = 2 \cdot 2024 \cdot 2025 + 2024 + 2025$$

positive divisors of n^2 that are less than n. All but one of the divisors of n are less than n. The number of these is $(2024 + 1) \cdot (2025 + 1) - 1 = 2024 \cdot 2025 + 2024 + 2025$. Hence there are

$$2 \cdot 2024 \cdot 2025 + 2024 + 2025 - (2024 \cdot 2025 + 2024 + 2025) = 2024 \cdot 2025 = 4{,}098{,}600$$

divisors of n^2 that are less than n but do not divide n.

5. Ten boxes with $1, 2, 3, \ldots, 9$, and 10 marbles are set on the table. Two players are taking turns picking one marble from any one of the boxes. The game ends when there are 3 marbles left. If the 3 marbles are in 3 different boxes, then the second player wins. Otherwise, the first player wins. Does one of the players have a winning strategy?

SOLUTION. The first player has a winning strategy. To win, the first player can follow the following two rules at each of the turns:

a) If there is a box with one marble, pick that marble (or one of such marbles if there are several such boxes).

b) Never pick a marble from a box with two marbles.

Apart from these rules, the first player can make any move.

Notice that it will always be possible to follow these rules. Indeed, the only situation when this would not be possible is when all the remaining boxes have exactly two marbles. But that would mean that the total number of marbles left in the boxes is even. This cannot happen because the number of marbles left before the first player's turn is always odd because the initial number of marbles is 55 and after each player completes their turn, the number decreases by 2, thus remaining odd.

Let us now show that by following these rules the first player always wins. Notice that after the first player completes their turn, there will never be a box with one marble. This will definitely be true after the first turn as the first player will pick up the marble from the first box, following rule a). After that, if before the second player's turn there are no boxes with one marble, the second player can produce no more than one box with one marble. That single marble will then be picked up by the first player on the next move, and so on. The first player will never produce a box with a single marble because of rule b).

We start with an odd number of marbles, and therefore the final three marbles will be left before the first player's turn. Since there will be no more than one box with one marble, as explained above, the first player wins.

Note that this is not the only possible winning strategy for the first player.