WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET I (2024–2025)

1. The sum of three integers is equal to 1, and their product is equal to 24. What can these three integers be?

SOLUTION 1. None of the integers can be equal to zero, since in that case their product would be zero. They cannot all be positive because then their sum would be at least $1 + 1 + 1$ which is greater than 1. Hence we must have two negative and one positive integer.

The sum is an odd integer, so either all three of the integers are odd, or one is odd and two are even integers. If all three are odd, then the product would be odd as well. Hence, we have one odd and two even integers.

$24 = 2^3 \cdot 3$ has the following odd divisors: $1, -1, 3, -3$, one of these integers must appear among the three. Denote by $2a$ one of the even integers. Then we have the following possibilities for the three integers (since they add up to 1):

(a) $1, 2a, -2a$,

(b) $-1, 2a, 2 - 2a$,

(c) $3, 2a, -2 - 2a$,

(d) $-3, 2a, 4 - 2a$

The product has to be 24, which gives the following equations in the four cases:

(a) $-4a^2 = 24$, this has no integer (or even real) solutions since $-4a^2 \leq 0$.

(b) $(2a - 2)2a = 24$ which leads to $(a - 1)a = 6$. By considering the divisors of 6 (a must be a divisor of 6) we see that we have exactly two solutions: $a = 3$ and $a = -2$. These lead to the same triple: $-1, -4, 6$, which is a possible solution.

(c) $3 \cdot (2a)(-2 - 2a) = 24$ which leads to $a(a + 1) = -2$. By considering the divisors of 2 as possible values for a we get that there are no integer solutions.

(d) $(-3)(2a)(4 - 2a) = 24$ which leads to $a(a - 2) = 1$ which has no integer solutions since a can only be ± 1 and neither one is a solution.

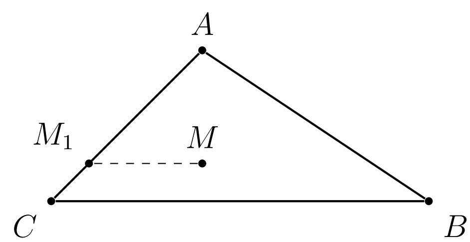
To summarize the only possible solution is $-1, -4, 6$.

SOLUTION 2. Notice that none of the integers is 0 since the product is not 0. Denote the numbers by a, b and c so that $|a| \geq |b| \geq |c| > 0$. First notice that all the numbers cannot be positive since their sum would be greater than 1. Since the product is positive, two of the numbers must be negative. If $a < 0$ then exactly one of b and c is positive, and hence $a + b + c$ would be negative, which cannot happen. This means that $a > 0$. Hence a can be $24, 12, 6, 4, 3, 2$ or 1. It is not difficult to see that if $a = 1$, $a = 2$ or $a = 3$ then $|a|$ cannot be largest. If $a = 4$ then, for the product to be equal to 24 (with $4 \geq |b| \geq |c|$), we must have $b = -3$, $c = -2$ and the sum is -1. If $a = 24$, then $b = c = -1$ and the sum is 22. If $a = 12$ then $b = -2, c = -1$ and the sum is 9. Hence $a = 6$. The only possibilities for b and c are $b = c = -2$ or $b = -4$, $c = -1$. Out of these two, only the latter gives the right sum. Hence the only the solution is the triple $6, -4, -1$.

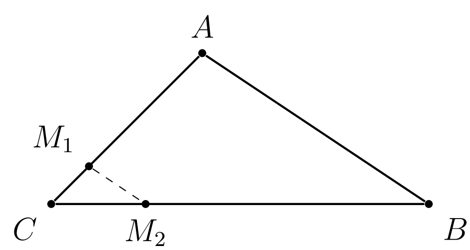
2. Point M located inside $\triangle ABC$ moves parallel to the side $\overline{BC}$ until it intersects with the side $\overline{CA}$, then parallel to $\overline{AB}$ until it intersects with $\overline{BC}$, then parallel to $\overline{AC}$ until it intersects with $\overline{AB}$, and so forth. Prove that, after some number of steps, the trajectory of the point will be closed.

SOLUTION. Let M_1 be the point on side $\overline{CA}$ where M lands when it moves parallel to the side $\overline{BC}$. Suppose

$$\frac{CM_1}{M_1A} = r.$$



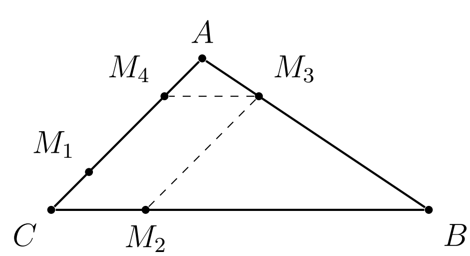
Similarly, let M_2 be the point on $\overline{BC}$ where the point lands when moving parallel to $\overline{AB}$.



By the intercept theorem (or similarity $\triangle CM_2M_1 \sim \triangle CBA$) ratios $CM2 : M_2B = CM_1 : M_1A$ are equal, so

$$\frac{CM_2}{M_2B} = \frac{CM_1}{M_1A} = r.$$

Now we continue this, creating point M_3 on $\overline{AB}$ and M_4 on $\overline{CA}$.



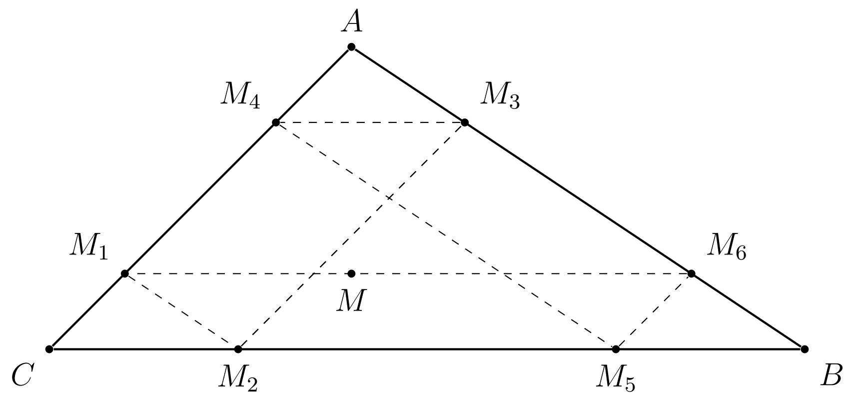
Similarly we can compute ratios:

$$r = \frac{CM_2}{M_2B} = \frac{AM_3}{M_3B} = \frac{AM_4}{M_4C},$$

so

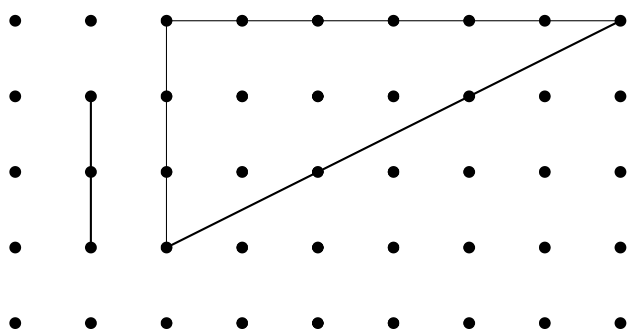
$$\frac{CM_4}{M_4A} = \frac{1}{r}.$$

It means that when point moves around the triangle ($M_1 \to M_2 \to M_3 \to M_4$) ratio on the side changes to its reciprocal. When we will do it one more time ($M_4 \to M_5 \to M_6 \to M_7$) trajectory will close.

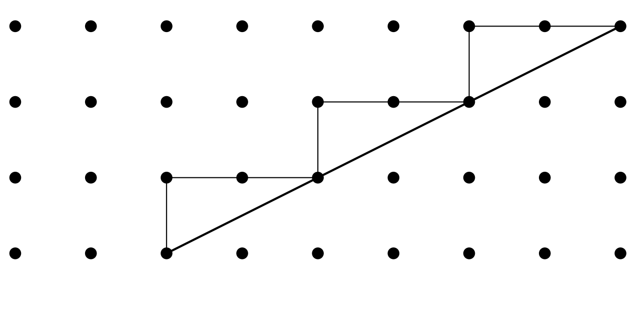


3. We draw a 2024×2024 grid of unit squares. We call the vertices of the unit squares in the grid *lattice points*; there are 2025^2 of these in our grid. Someone chose 10 of these lattice points, and drew all the line segments that connect any two of them. Show that at least one of these drawn line segments will contain at least two more lattice points besides its end points.

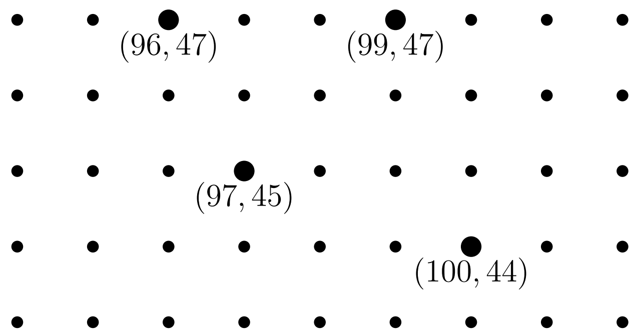
SOLUTION. When will the line segment connecting two lattice points contain additional lattice points? If the two lattice points are in the same row or column, and they are not next to each other, then that is certainly the case. If they are not in the same row and column, we can form a right-angled triangle with these points as vertices of the hypotenuse, and the two other sides parallel to the sides of the grid. The side lengths of the legs of the triangle are integers since these are horizontal and vertical line segments connecting lattice points.



Denote the horizontal and vertical side lengths by x and y, respectively. If these numbers have a common integer divisor d, then the original line segment will contain at least $d-1$ additional lattice points. We can find these by starting at one of the original lattice points, and stepping x/d units horizontally and y/d vertically towards the other point. After each such step we will be at a lattice point since x/d and y/d are integers, and we will stay on the same line segment since the corresponding right angle triangle will be similar, and hence, have the same angles and the original one.

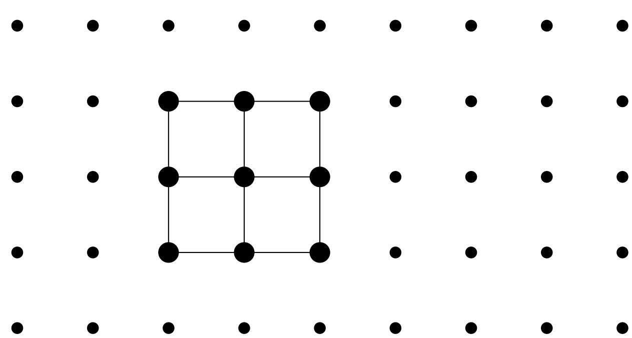


Let's get back to our problem. Each lattice point in the grid can be identified by two integers between 0 and 2024: the distances from the bottom and the left side of the grid. We call these the *coordinates* of our lattice point. See the picture below for explanation.



For each lattice point coordinate consider the remainder after division by 3: this can be 0, 1, 2 for each of the two numbers. For the two coordinates we can have $3 \cdot 3 = 9$ different selections for the two remainders. Because we have 10 lattice points, we will have at least two lattice points, say A and B, where the two remainders agree. But that means that the difference of the first coordinates is divisible by 3, and the same is true for the difference of the second coordinates. If A and B are in the same row or column, then their distance is at least 3, and hence the line segment connecting them will contain at least two more lattice points. If they are not in the same row or column, then we can construct a right angle triangle as above, where $\overline{AB}$ is the hypotenuse, and the two legs are parallel to the sides of the grid and are both multiples of 3. But since both sides are divisible by 3, we must have at least two more lattice points on the line segment connecting them, as in the second diagram.

Note that the statement is sharp: if we choose 9 lattice points in the form of a 3×3 grid, then none of the connecting line segments will have at least two more lattice points on them.



4. Let a, b, c be positive real numbers such that $abc = 1$. Show that if

$$a + b + c > \frac{1}{a} + \frac{1}{b} + \frac{1}{c},$$

then exactly one of the three numbers is greater than 1.

SOLUTION 1.

Let us rewrite the condition as

$$a + b - \left(\frac{1}{a} + \frac{1}{b}\right) > \frac{1}{c} - c = \frac{1 - c^2}{c} = \frac{(1 + c)(1 - c)}{c}.$$

Now notice that

$$\frac{1}{a}+\frac{1}{b}=\frac{a+b}{ab}=(a+b)c$$

since we know that $c=\frac{1}{ab}$. Hence, the given inequality becomes

$$(a+b)(1-c)>(1-c)\frac{1+c}{c}.$$

Bringing all the terms to the left side, we see that

$$\left(a+b-\frac{1+c}{c}\right)(1-c)>0,$$

while using the identity $c=\frac{1}{ab}$ again shows that the right side factors as

$$\left(a+b-\frac{1+c}{c}\right)(1-c)=\left(a+b-\frac{1}{c}-1\right)(1-c)=(a+b-ab-1)(1-c)=-(1-a)(1-b)(1-c).$$

This expression is positive when either one or all three factors $(1-a),(1-b),(1-c)$ are negative. The former case is what is claimed, while the latter is impossible, as in this case we would have $a,b,c>1$, which would contradict $abc=1$.

SOLUTION 2. Notice that

$$\begin{aligned}(a-1)(b-1)(c-1)&=abc-ab-bc-ac+a+b+c-1\\&=1-\frac{1}{c}-\frac{1}{a}-\frac{1}{b}+a+b+c-1\\&=a+b+c-\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)>0.\end{aligned}$$

Therefore, either exactly one of the three expressions $a-1$, $b-1$, $c-1$ is positive, as claimed, or all three are positive. However, $a>1,b>1,c>1$ is not possible because of $abc=1$.

5. Let $n>1$ be an integer. We wrote the numbers 1, 2, 2^2, $\dots 2^{n-1}$ on a board, and circled the first digit of each number. Show that out of the nine possible digits 1, 2, …, 9 there is at least one that is circled at most $\frac{n}{17}$ times.

SOLUTION. Fix the value of n. For $i=1,2,\dots,9$ let a_i denote the number of times the digit i is circled on the board.

The number 2^k has first digit i exactly if there is a positive integer m with

$$i\cdot 10^{m-1}\le 2^k<(i+1)\cdot 10^{m-1}.$$

Indeed: the least m digit positive integer with first digit i is $i\cdot 10^{m-1}$, and the greatest one is $(i+1)10^{m-1}-1$. Hence if m the number of digits of 2^k then the inequality holds.

Suppose that $1\le k$, and the first digit of 2^k is 2. Then for some $m\ge 1$ we have

$$2\cdot 10^{m-1}\le 2^k<3\cdot 10^{m-1},$$

and dividing both sides by 2 gives

$$1\cdot 10^{m-1}\le 2^{k-1}<1.5\cdot 10^{m-1}<2\cdot 10^{m-1}$$

showing that 2^{k-1} must have a first digit equal to 1. Similarly, if the first digit of 2^k is 3, then we have (for some m)

$$1 \cdot 10^{m-1} \leq 1.5 \cdot 10^{m-1} \leq 2^{k-1} < 2 \cdot 10^{m-1},$$

so the first digit of 2^{k-1} has to be 1 in this case as well.

This means that any time we circle a 2 or a 3, the previous number in the sequence $1, 2, \ldots, 2^{n-1}$ must have a 1 circled as a first digit. In particular, we have

$$a_2 + a_3 \leq a_1.$$

We do not have an equality because 2^{n-1} might start with the digit 1. We also used the fact that in our sequence the first number does not have a first digit that is 2 or 3.

The same argument shows that if $k \geq 1$ and 2^k has a first digit 4 or 5, then 2^{k-1} has a first digit 2, leading to

$$a_5 + a_4 \leq a_2,$$

and repeating this argument for 6 and 7, and then for 8 and 9 gives

$$a_6 + a_7 \leq a_3,$$

and

$$a_8 + a_9 \leq a_4.$$

Let m be the least of the numbers $a_1, \ldots, a_9$. This means that $m \leq a_i$ for $i = 1, \ldots, 9$. Using our inequalities this also leads to

$$\begin{aligned}
a_4 &\geq a_8 + a_9 \geq m + m = 2m,\\
a_3 &\geq a_6 + a_7 \geq m + m = 2m,\\
a_2 &\geq a_4 + a_5 \geq 2m + m = 3m,\\
a_1 &\geq a_2 + a_3 \geq 3m + 2m = 5m.
\end{aligned}$$

Collecting all of our bounds gives

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 \geq 5m + 3m + 2m + 2m + m + m + m + m + m = 17m.$$

Since the $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9$ is the total number of digits we circled, we have

$$n = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 \geq 17m$$

leading to $\frac{n}{17} \geq m$. This shows that the digit that is circled the fewest number of times was circled at most $\frac{n}{17}$ times.

Remark. It can be shown that for each digit $1, \ldots, 9$ the ratio of times it was circled among the n numbers approaches a limiting value as n grows greater and greater. The following table shows the (approximate) asymptotic ratios of the circled digits:

1	2	3	4	5	6	7	8	9
0.30103	0.176091	0.124939	0.09691	0.0791812	0.0669468	0.0579919	0.0511525	0.0457575

Note that $\frac{1}{17} \approx 0.0588235$.