WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET V (2023-2024)

1. Initially, each entry in a 4×4 table is equal to 0. In each step we can choose a 2×2 subtable and increase each of the four numbers in that subtable by one. Decide if the following table can be obtained in a finite number of steps.

3	4	3	2
6	11	11	4
4	12	11	5
1	3	5	3

SOLUTION. We will show that the table cannot be obtained.

Let us label the squares in the table as follows:

a	b	c	d
e	f	g	h
i	j	k	ℓ
m	n	o	p

In each step we are increasing the numbers in one of nine possible 2×2 subtables. There are 4 corner ones (e.g. the one in the top left containing a, b, e, f), there are 4 near the middle of an edge (e.g. the one containing b, c, f, g), and there is one in the middle (containing f, g, j, k).

In order to obtain the given table, the top left corner number can only change if we perform a step using the top left corner 2×2 table, which means that we had to use that step exactly 3 times. Similarly, the subtable containing c, d, g, h was used twice, the subtable containing k, ℓ, o, p was used 3 times, and the subtable i, j, m, n was used once. If we remove the effects of these moves from the given table, we get

0	1	1	0
3	8	9	2
3	11	8	2
0	2	2	0

If the original table could be obtained, then this one has to be obtained without using any of the four corner moves.

The number in position b can only be changed by the corner move a, b, e, f or the edge move b, c, f, g. Hence, if the previous table was obtainable without corner moves, then we had to use the edge move a, b, e, f exactly once. Similarly, we had to use the edge move g, h, k, ℓ exactly twice, the move j, k, n, o exactly twice, and the move e, f, i, j exactly three times. Removing the effects of these moves gives

0	0	0	0
0	4	6	0
0	6	4	0
0	0	0	0

If the original configuration can be obtained, then this one can be obtained with just using the middle move. Since the numbers in the positions f, g, j, k are not equal to each other, this is not possible.

ALTERNATE SOLUTION. Here is a quicker solution. Suppose that we color the squares in the table with a checker-board pattern: a, c, f, h, i, k, n, p all white and the others black. Then each 2×2 subtable contains exactly two black and two white squares, which means that with each move the sum of the numbers in the white squares increases by 2, and the sum of the numbers in the black squares increases by 2. Since these sums are zero at the beginning, the two sums must be the same after any number of moves. If we add the the numbers in the white squares for the given configuration, we get $3+3+11+4+4+11+3+3 = 42$. For the black squares the sum is $4+2+6+11+12+5+1+5 = 46$. Since these numbers are not equal to each other, we cannot obtain the table using the given steps.

2. Maya wrote down a power of two on the board. Ian wrote down a different number by rearranging the digits of Maya's number. Show that Ian's number cannot be a power of two. (Ian cannot move a zero digit to the first position.)

SOLUTION. Let Maya's number be 2^a for $a \geq 0$, and suppose that after rearranging digits, Ian obtained a different power of two, say, 2^b for some $b \geq 0$, $b \neq a$. Let us show that this is impossible.

Indeed, suppose $a < b$. Since $2^b = (2^{b-a})2^a$ and 2^a differ by a factor of 2^{b-a} and have the same number of digits, $2^{b-a} < 10$. This gives three options: $b - a = 1, 2, 3$, that is, $2^{b-a} = 2, 4, 8$. Also, we know that for any number, its remainder when divided by 9 equals the remainder of the sum of its digits. Since 2^a and 2^b have the same sum of digits, they leave the same remainder when divided by 9, and therefore, $2^b - 2^a$ is divisible by 9. However, we have

$$(2^b - 2^a) = 2^a(2^{b-a} - 1),$$

where the second factor is $2^1 - 1 = 1$, $2^2 - 1 = 3$, or $2^3 - 1 = 7$. We therefore see that the prime factorization of $2^b - 2^a$ cannot possibly contain two 3's: it contains at most one when $b - a = 2$. At any rate, it cannot possibly be divisible by $9 = 3^2$.

The case $b < a$ is similar with a and b interchanged.

Remark. Another approach is to show that the remainders when powers of 2 are divided by 9 follow the pattern 1, 2, 4, 8, 7, 5, 1, 2, 4, 8, 7, 5, ..., so it is impossible for two powers 2^a and 2^b to have the same remainder if their exponents $a \neq b$ differ by no more than 3.

3. We have a geometric progression of n positive integers with $n \geq 2$. (This means that the ratio of each two consecutive integers in the progression is the same.) Show that the average of all the n terms in the progression cannot be greater than the average of the first and last term of the progression.

SOLUTION. Let a be the first element of the geometric progression, and let q be the ratio of two successive terms. Then the n elements of the progression are a, aq, aq^2, ..., aq^{n-1}. The kth term is aq^{k-1} where $k = 1, 2, \ldots, n$.

We will show that for any $1 \leq k \leq n$ the sum of the kth and $(n+1-k)$th terms is at most as large as the sum of the first and last terms:

$$aq^{k-1} + aq^{n-k} \leq a + aq^{n-1}.$$

Indeed, rearranging the terms to the right we get

$$0 \leq a - aq^{k-1} + aq^{n-1} - aq^{n-k} = a(1 - q^{k-1}) + aq^{n-k}(q^{k-1} - 1) = a(1 - q^{k-1})(1 - q^{n-k}).$$

Since the terms in the progression are positive, we must have $a > 0$ and $q > 0$. If $q = 1$, then $a(1 - q^{k-1})(1 - q^{n-k}) = 0$. If $0 < q < 1$, then both $1 - q^{k-1}$ and $1 - q^{n-k}$ are between 0 and 1, so

$a(1-q^{k-1})(1-q^{n-k}) \geq 0$. Finally, if $q > 1$, then $q^{k-1}-1$ and $q^{n-k}-1$ are both at least 0; hence, $a(1-q^{k-1})(1-q^{n-k}) = a(q^{k-1}-1)(q^{n-k}-1) \geq 0$. This proves the inequality in all cases.

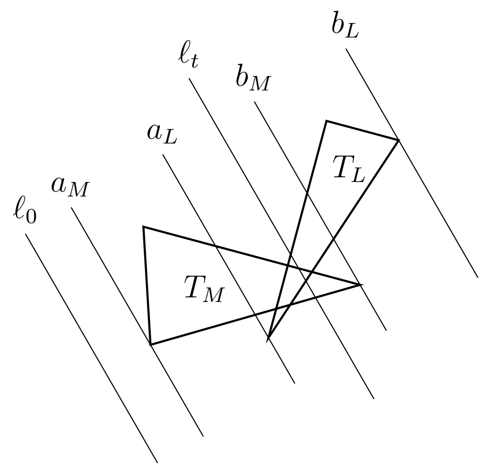
Now add up the inequality we just proved for $k = 1, 2, \ldots, n$ to get

$$2(a + aq + aq^2 + \cdots + aq^{n-1}) \leq n(a + aq^{n-1}).$$

Dividing both sides by $2n$ shows that the average of the n terms is at most as large as the average of the first and last terms.

4. There are 2024 triangles in the plane so that any two of them intersect with each other. Show that we can draw a straight line that intersects all the triangles.

SOLUTION. Let the triangles be denoted by T_1, T_2, T_3, $\ldots$, T_{2024} and let ℓ_0 be a line in the same plane such that all the triangles lie in one of the half-planes bounded by ℓ_0. Let ℓ_t, $t > 0$ be the line obtained from ℓ_0 via parallel translation by t units in the direction perpendicular to ℓ_0 into the half-plane containing the triangles. Let the intervals $[a_k, b_k]$, $k = 1, 2, \ldots, 2024$, be the sets of t such that ℓ_t intersects T_k. Let a_L be the greatest of the a_k, $k = 1, 2, \ldots, 2024$ and let b_M be the least of the b_k, $k = 1, 2, \ldots, 2024$. Notice that $a_L \leq b_M$. Indeed, by the conditions of the problem, the triangles T_L and T_M must intersect. Hence, there must exist a line ℓ_t passing through a point of intersection of T_L and T_M. Then t belongs to the intersection of the intervals $[a_L, b_L]$ and $[a_M, b_M]$ implying that $a_L \leq b_M$.



Therefore, $a_L \leq b_M$, and we can pick t such that $a_L \leq t \leq b_M$. Notice that then, by our choice of a_L and b_M, $a_k \leq t \leq b_k$ for all $k = 1, 2, \ldots, 2024$. Thus, the line ℓ_t intersects all of the triangles.

5. A baker baked a rectangular pie and cut it into n^2 rectangles by making $n-1$ vertical cuts and $n-1$ horizontal cuts. (n is at least 2.) The areas of the resulting pie pieces rounded to the nearest integers are equal to all the natural numbers from 1 to n^2 in some order. What is the greatest n for which this is possible? (Semi-integer numbers are rounded upward.)

SOLUTION. We show that the described situation can be achieved for $n = 4$, but not for any $n > 4$.

The following table shows a possible solution for $n = 4$ with a rectangular pie of size 9.65×14. The 16 pie pieces have areas 1, 2, $\ldots$, 16 after rounding.

	2	3	4	5
0.7	$1.4 \approx 1$	$2.1 \approx 2$	$2.8 \approx 3$	$3.5 \approx 4$
2.7	$5.4 \approx 5$	$8.1 \approx 8$	$10.8 \approx 11$	$13.5 \approx 14$
3	6	9	12	15
3.25	$6.5 \approx 7$	$9.75 \approx 10$	13	$16.25 \approx 16$

Next we prove that we must have $n \leq 4$. Let's rearrange the rows and columns of the table so that the heights of the rows grow from top to bottom, and column widths grow from left to right. Let the smallest of the rectangular pieces have dimensions $x_{min} \times y_{min}$, and the the largest have dimensions $x_{max} \times y_{max}$. Let the numbers in the four corner cells be $a < b < c < d$. Then $a = x_{min} \cdot y_{min}$ is the upper-left number, $d = x_{max} \cdot y_{max}$ the lower right number, and we have $ad = bc = x_{min} \cdot y_{min} \cdot x_{max} \cdot y_{max}$. Assume the upper right number is $b = x_{max} \cdot y_{min}$. (The other case can be handled similarly.)

Denote the rounded version of a number z by z'. Then we must have $a' = 1$ and $d' = n^2$ since these are the least and the greatest of the first n^2 integers. We also have $b' \geq n$ since b' cannot be less than the other numbers of the top row. Finally, we must have $c' \geq 2n - 1$ since c' cannot be less than any of the other numbers of the first column and the top row.

From $a' = 1$ and $d' = n^2$ we get $a < 1.5$ and $d < n^2 + 0.5$. From the bounds $b' \geq n$ and $c' \geq 2n - 1$ we get $b \geq n - 0.5$ and $c \geq 2n - 1.5$. Therefore,

$$1.5(n^2 + 0.5) > ad = bc > (n - 0.5)(2n - 1.5).$$

Expanding the two sides and simplifying gives $1.5n^2 + 0.75 > 2n^2 - 2.5n + 0.75$ and $2.5n > 0.5n^2$, which implies $n < 5$ and, hence, $n \leq 4$.