WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET IV (2023-2024)

1. A sequence of numbers is formed according to the following rules: in the first place, we put "1", and, if the number in the k-th place is N, then we put $2N+1$ in the $(2k)$-th place and $2N$ in the $(2k+1)$-st place. (So the first three numbers of the sequence are $1, 3, 2$.) Show that every positive integer will eventually appear in this sequence, but only once.

SOLUTION. *First approach.* It follows from the rules that any number appearing in the even place is going to be odd, and any number appearing in the odd place is even. (With the exception of 1 in the first place.) It is also clear that any number past the first is greater than 1.

Let N be any positive integer. If $N = 1$, it appears in the first place, and only in the first place, so suppose $N > 1$. If N is odd, it can only appear in an even place, say, in the $(2k)$-th place, and this will happen exactly when $\frac{N-1}{2}$ appears in the k-th place. So, we see that N appears exactly once as long as $\frac{N-1}{2}$ appears exactly once.

If N is even, the situation is similar: it can only appear in an odd place, say, in the $(2k+1)$-st place, and this will happen exactly when $\frac{N}{2}$ appears in the k-th place. So, N appears exactly once as long as $\frac{N}{2}$ appears exactly once.

In either case, we see that in order to show that N appears exactly once, we need to check that a smaller positive number (either $\frac{N-1}{2}$ or $\frac{N}{2}$ appears exactly once. Repeating the process, we will eventually reduce our statement to the case of number 1 appearing exactly once, which we already know.

Second approach. Let us separate the sequence into segments by powers of 2: n-th segment consists of the numbers in places between 2^{n-1} and $2^n - 1$. We will show that each segment contains the numbers between 2^{n-1} and $2^n - 1$, but arranged in the descending order:

$$1, \quad 3, 2, \quad 7, 6, 5, 4, \quad 15, 14, 13, 12, 11, 10, 9, 8, \quad 31, 30, \ldots$$

This implies the statement.

We will prove the statement by induction. The first segment consists of the number 1, as we are told. Suppose that the n-th segment is as claimed: explicitly, that for any k such that $2^{n-1} \leq k \leq 2^n - 1$, the number in the k-th place is $2^n - 1 + 2^{n-1} - k$. Then in the $(2k)$-th place we put

$$2 \times (2^n - 1 + 2^{n-1} - k) + 1 = 2^{n+1} - 1 + 2^n - (2k),$$

and in the $(2k+1)$-st place, we put

$$2 \times (2^n - 1 + 2^{n-1} - k) = 2^{n+1} - 1 + 2^n - (2k+1).$$

In either case, we see that the formula for the number in K-th place (for $K = 2k$ or $K = 2k+1$) is $2^{n+1} - 1 + 2^n - K$. As k goes from 2^{n-1} to $2^n - 1$, the quantities $2k$ and $2k+1$ sweep all the numbers between 2^n and $2^{n+1} - 1$, therefore, the formula holds for all $2^n \leq K \leq 2^{n+1} - 1$. As K goes over this interval, the formula describes the descending sequence from $2^{n+1} - 1$ to 2^n, as claimed.

2. Is it true that for any $x \geq 1$ the following statement holds?

$$\lfloor \sqrt{x} \rfloor = \left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor$$

Recall the definition of $\lfloor y \rfloor$: for a real number y this is the *floor* of y, the greatest integer less than or equal to y.

SOLUTION. Yes, it is true. Let $1 \leq x$ and $n = \lfloor\sqrt{x}\rfloor$. Then because n and $n+1$ are integers

$$\begin{aligned} n &\leq \sqrt{x} < n+1 \\ n^2 &\leq x < (n+1)^2 \\ n^2 &\leq \lfloor x \rfloor \leq x < (n+1)^2 \\ n &\leq \sqrt{\lfloor x \rfloor} < n+1. \end{aligned}$$

Hence, $\lfloor\sqrt{x}\rfloor \leq \left\lfloor\sqrt{\lfloor x\rfloor}\right\rfloor < \lfloor\sqrt{x}\rfloor + 1$. Therefore, $\lfloor\sqrt{x}\rfloor = \left\lfloor\sqrt{\lfloor x\rfloor}\right\rfloor$.

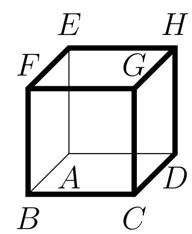
3. Call a positive integer *nice* if all of its digits are from 2, 3, 4, 5, 6, 7. How many 100-digit nice numbers are multiples of 2^{100}?

SOLUTION. We show that the answer is 3^{100}. Let's prove by induction that there are exactly 3^n good n-digit numbers (divisible by 2^n and composed of the given digits). The base case ($n = 1$) is true: exactly three of the single digit nice numbers are even: 2, 4, and 6.

Now consider the induction step. Assume that the statement is true for some n, we would like to prove that it is also true for $n+1$. If we erase the first digit of a good $(n+1)$-digit number, we get a good n-digit number. Indeed, by erasing the digit x, we subtract number $x \cdot 10^n$ (divisible by 2^n) from the number divisible by 2^{n+1}.

On the other hand, a good n-digit number has the form $y \cdot 2^n$ with an integer y. By adding the digit x to the left-hand side, we add the number $10^n \cdot x = (x\dot{5}^n)2^n$ and the sum will be divisible by 2^{n+1} if and only if the number $y + x \cdot 5^n$ is even, that is, when $x + y$ is even. This happens exactly for three possible digits x: if y is even then x has to be even (so one of 2, 4, 6), and if y is odd then x has to be odd (one of 3, 5, 7). So there are 3 times as many good $(n+1)$-digit numbers as there are good n-digit numbers, proving the induction step and the statement.

4. Consider a unit cube with vertices $ABCDEFGH$, as shown in the picture. Identify the set of points on the face $ABCD$ which are of equal distance from the vertices A and G.



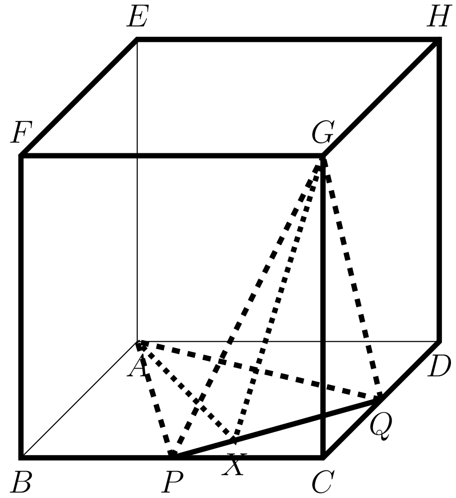
SOLUTION. Let P be the midpoint of the edge $\overline{BC}$ and Q the midpoint of the edge $\overline{CD}$. We will show that the line segment $\overline{PQ}$ is the set of points on the face $ABCD$ that are of equal distance from A and G.

The length of the line segment $\overline{AP}$ can be computed using the right-angled triangle ABP with the Pythagorean theorem:

$$AP = \sqrt{AB^2 + BP^2} = \sqrt{5/4}.$$

A similar calculation from the GCP right-angled triangle gives $GP = \sqrt{GC^2 + CP^2} = \sqrt{5/4}$ showing $AP = GP$. The same argument shows $AQ = GQ$.

Now take a point X on the line segment $\overline{AP}$. The triangles APQ and GPQ are congruent because $AP = GP$ and $AQ = GQ$. Then the angles $\angle APQ$ and $\angle GPQ$ are also equal, which shows that the triangles APX and GPX are congruent. But this means $AX = GX$. Hence all the points of PQ satisfy the requirements, we just need to show that there are no other points on the face $ABCD$ that are of equal distance from A and G.



The points in the space that are of equal distance from the points A and G are on the plane that is perpendicular to the line segment AG and intersects it at its midpoint. Let us denote this plane by h. The set we are interested in is the intersection of the plane h with the face $ABCD$. The intersection of two distinct planes is either the empty set (if they are parallel), or a line. Let the intersection of the plane of $ABCD$ with the plane h denoted by ℓ. The set we are interested in is the intersection of ℓ with the face $ABCD$. Since we showed that the set contains the line segment PQ, this means that ℓ has to be the line going through P and Q. But the intersection of this line with the face $ABCD$ is exactly the line segment PQ, which shows that the points of this line segments are the only ones on the face $ABCD$ that are of equal distance from the points A and G.

5. A warehouse stores 100 boxes containing nuts, bolts, and washers. Each box contains at least one nut, one bolt, and one washer. Prove that one can choose 51 boxes so that together they contain more than half of all the nuts, more than half of all the bolts, and more than half of all the washers.

SOLUTION. Choose a box with the largest number of nuts (let this be n^*) and set it aside. From the remaining boxes, choose the box with the largest number of bolts (let this be b^*) and set it aside. Let n_k and b_k, $n = 1, 2, ..., 98$, be the numbers of nuts and bolts in the remaining boxes enumerated in such a way that n_k are non-increasing:

$$n^* \geq n_1 \geq n_2 \geq \cdots \geq n_{98}.$$

We also have $b^* \geq b_k$ for all $k = 1, \ldots, 98$.

We will show that we can split the remaining 98 boxes into two groups of 49 so that the following conditions are satisfied:

(1) the difference of nuts between the two groups is at most n^*, and

(2) the difference of bolts between the two groups is at most b^*.

Suppose that we can achieve such an arrangement, we will show that the statement of the problem follows. Now choose the group of 49 with the larger total number of washers, add the originally set aside two boxes. We show that the resulting 51 boxes will satisfy the requirements. (If the two groups have the same total number of washers, then we can choose either one.) The chosen group of 49 boxes have at least as many total washers as the other 49. After adding the two additional boxes (which have at least two washers in them) the resulting 51 chosen boxes will have more washers than the other 49, hence they have more than half the total washers. By (1) the difference of nuts between the two groups of 49 is at most n^*, so when we add the two additional boxes (which have at least $n^* + 1$ nuts in them) to either one of the groups we will have more than half of all the nuts there. The same argument shows (using (2)) that the 51 chosen boxes will have more than half of all the bolts. Hence we just have to show that we can create the two groups of 98 satisfying conditions (1) and (2).

We first create two groups of 49 as follows: odd numbered boxes go into the first group, even numbered into the second. Then the difference between the total number of nuts between the two groups is

$$(n_1 - n_2) + (n_3 - n_4) + \cdots + (n_{97} - n_{98}). \qquad (*)$$

Since we have $n^* \geq n_1 \geq n_2 \geq \cdots \geq n_{98}$ by assumption, we have

$$\begin{aligned} 0 \leq (n_1 - n_2) + (n_3 - n_4) + \cdots + (n_{97} - n_{98}) &= n_1 + (n_3 - n_2) + (n_5 - n_4) + \cdots + (n_{97} - n_{96}) - n_{98} \\ &\leq n_1 - n_{98} \leq n_1 \leq n^*. \end{aligned}$$

This shows that the expression given in $(*)$ is non-negative and it is at most n^*.

If the difference between the total number of bolts in the two groups is at most b^* then we are done. Let us assume that the difference in the total number of bolts in the two groups is greater than b^*. We change the groups in the following way. Assume that there are more bolts in the first group. (The same approach works in the other case as well.) Then for some j, $b_{2j-1} > b_{2j}$. Switch those boxes between the groups, then the difference of bolts between the two groups changed by $2(b_{2j-1} - b_j) \leq 2b^*$. After this change the difference of bolts between the first group and second group is either

(a) positive, but still greater than b^*,

(b) non-negative, but at most b^*,

(c) negative, but greater than $-b^*$ (since it was larger than b^* originally, and we decreased it by at most $2b^*$).

In the cases (b) and (c) we stop. In the case (a) we continue the switching: since the difference is positive, we can find another j with $b_{2j-1} > b_{2j}$, and we can switch the corresponding boxes. We repeat this until we get to case (b) or case (c). (Since there are only 49 pairs we can switch, eventually this have to happen.) When we stop, the difference of bolts between the first and second group is between $-b^*$ and b^*, as required. What happened to the difference of nuts between the two groups? At the beginning the difference was given by the expression $(*)$. In $(*)$ each term $n_{2j-1} - n_{2j}$ is non-negative since $n_{2j-1} \geq n_{2j}$. After the switching, the terms corresponding to the switched pairs are multiplied by -1, these terms become non-positive, but their absolute value will not change. But that means that the *absolute value* of the difference cannot increase. This follows from the fact that the absolute value of a sum is always at most as large as the sum of the absolute values. Here is a more detailed justification: after the switching $(*)$ becomes

$$x_1(n_1 - n_2) + x_2(n_3 - n_4) + \cdots + x_{49}(n_{97} - n_{98}),$$

where $x_j = 1$ (if the jth pair was not switched) or $x_j = -1$ (if the pair was switched). Hence

$$\begin{aligned} |x_1(n_1 - n_2) + x_2(n_3 - n_4) + \cdots + x_{49}(n_{97} - n_{98})| &\leq |x_1(n_1 - n_2)| + |x_2(n_3 - n_4)| + \cdots + |x_{49}(n_{97} - n_{98})| \\ &= |n_1 - n_2| + |n_3 - n_4| + \cdots + |n_{97} - n_{98}| \\ &= (n_1 - n_2) + (n_3 - n_4) + \cdots + (n_{97} - n_{98}) \leq n^*. \end{aligned}$$

This shows that after our switching procedure conditions (1) will still be satisfied, which means that the statement follows.