WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET III (2023-2024)

1. How many positive integers can you find so that the 3rd largest divisor of the number is equal to 2024? (We count the number itself as a divisor.)

SOLUTION. Suppose that n is a positive integer with 2024 as its 3rd largest divisor. The largest divisor of n is n itself. All other divisors of n are of the form n/k with $k > 1$ being a positive integer. The larger k is the smaller n/k will be.

Since 2024 is divisible by 4, n has to be divisible by 4 as well. That means that $n/2$ and $n/4$ are both integers, these are also divisors of n. Hence, $n/2$ is the second largest divisor of n. If n is divisible by 3, then the third largest divisor of n is $n/3$, if n is not divisible by 3, then the third largest divisor is $n/4$. Hence, we have $n/3 = 2024$ and $n = 6072$, or $n/4 = 2024$ and $n = 8096$. (In the second case n is not divisible by 3, so 2024 is indeed the third largest divisor.) Hence, there are two integers satisfying the conditions of the problem: 6072 and 8096.

2. An arithmetic progression is an infinite sequence of numbers where the difference between each two consecutive terms is the same. A geometric progression is an infinite sequence of non-zero numbers where the ratio of each two consecutive terms is the same. We found that for a certain arithmetic progression the 1st, 2nd and 2024th elements give the first three numbers of a certain geometric progression (in that order). Show that all the elements of this geometric progression must appear somewhere in our arithmetic progression.

SOLUTION. Denote by $a_1, a_2, \dots$ the elements of the arithmetic progression and with $g_1, g_2, \dots$ the elements of the geometric progression. Then we have $g_1 = a_1$, $g_2 = a_2$, and $g_3 = a_{2024}$.

Denote the difference between each two consecutive terms of the progression $a_1, a_2, \dots$ by D. Then $g_2 - g_1 = a_2 - a_1 = D$ and $g_3 - g_2 = a_{2024} - a_2 = 2022D$. Let R be the ratio of each two consecutive terms in the geometric progression:

$$R = \frac{g_2}{g_1} = \frac{g_3}{g_2} = \frac{g_4}{g_3} = \dots$$

If $R = 1$ then all elements of the geometric progression are equal, which would also imply $a_1 = a_2$ and $D = 0$. In this case the statement of the problem holds (since the two sequences are the same.)

If $R \neq 1$ then the subsequent elements of the geometric sequence cannot be equal to each other, and for any k,

$$g_{k+1} - g_k = g_k R - g_{k-1} R,$$

so

$$R = \frac{g_{k+1} - g_k}{g_k - g_{k-1}}.$$

In particular, for $k = 2$ we get

$$R = \frac{g_3 - g_2}{g_2 - g_1} = \frac{2022D}{D} = 2022.$$

Hence, $a_1 + D = a_2 = g_2 = 2022g_1 = 2022a_1$, which gives us $D = 2021a_1$, and the two progressions are of the form

$$a_1, (1 + 2021)a_1, (1 + 2021 \times 2)a_1, \dots, a_k = (1 + 2021(k-1))a_1, \dots$$

and

$$a_1, 2022a_1, 2022^2 a_1, \ldots, g_k = 2022^{k-1} a_1, \ldots.$$

It now remains to notice that, for any k, 2022^{k-1} leaves remainder 1 when divided by 2021, and, therefore, it is of the form $1 + 2021(N-1)$ for some $N > 0$ (and then $g_k = a_N$).

3. The mn kids in the nursing room sit on mn soft square mats arranged in an $m \times n$ rectangle. Each kid is sitting on a different square, and each kid looks towards one of the sides of the rectangle (they might face in different directions). When the nanny claps her hands, each child crawls over to the next mat in the direction they were looking, and turns 90 degrees to the left or right. If a baby crawls off the mat onto the cold floor, it cries. If two babies end up on the same mat, they cry. For which values of m and n can these crawls last indefinitely without tears?

SOLUTION. We show that the babies can crawl around forever without crying exactly if m and n are both even.

If m and n are both even, then let's paint the mats in four colors as shown in in the figure.

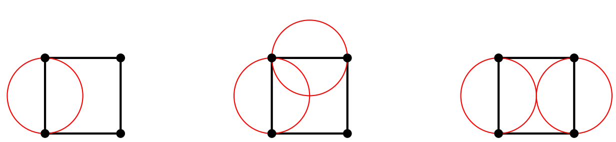
1	2	1	2	$\cdots$
3	4	3	4	$\cdots$
1	2	1	2	$\cdots$
3	4	3	4	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The rectangle is divided into 2×2 squares, and children can crawl inside the squares in a cycle: $1 \to 2 \to 3 \to 4 \to 1$.

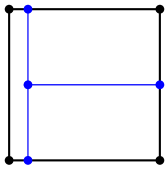
If at least one number m and n is odd, then with the same coloring the number of squares of color 1 is greater than the number of cells of color 4. Note that after two moves all the children from the mats of color 1 will be on the mats of color 4, or will leave the mat. Therefore, tears are inevitable.

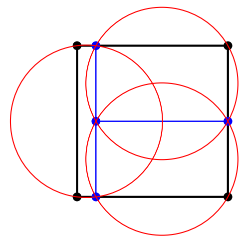
4. Show that a square with side length 1 cannot be fully covered with three identical disks of diameter 1, but it can be covered with three identical disks of diameter $\frac{101}{100}$.

SOLUTION. Suppose that three disks each with diameter 1 cover the unit square. The square has four vertices, so one of the disks must cover at least two of these vertices. If two vertices are on the same side of the square then their distance is 1, if they are the endpoint of a diagonal then their distance is $\sqrt{2} > 1$. Because the disk has diameter equal to the side length of the square, the two vertices covered by the same disk must be endpoints of a diameter of the disk, and must be on one of the sides of the square. That leaves three sides of the square that must be fully covered by the other two disks. Those two disks must cover all four vertices of the square, so, as before, one side of the square is the diameter of one of the disks. Whether the second disk covers a side of the square adjacent to the side that the first disk covers or it covers the side opposite the side that the first disk covers, it leaves two sides of the square uncovered. It is not possible for one disk with diameter 1 to cover those two remaining sides, so it is not possible to cover the square with three disks with diameter 1.



On the other hand, partition the square into three rectangles where one rectangle lies along a side of the square and has dimensions $1 \times \frac{1}{8}$, and the other two rectangles have dimensions $\frac{7}{8} \times \frac{1}{2}$, as shown.





By the Pythagorean theorem the first of these rectangles has a diagonal with length $\sqrt{1^2 + \left(\frac{1}{8}\right)^2} = \frac{\sqrt{65}}{8}$, and the other two rectangles have diagonals with length $\sqrt{\left(\frac{7}{8}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{65}}{8}$. A rectangle can be covered with a disk with diameter equal to the diagonal (using the intersection of the diagonals as the center of the disk). Thus, each of these three rectangles can be covered by a disk with diameter $\frac{\sqrt{65}}{8} \approx 1.00778$, which is less than $\frac{101}{100} = 1.01$ as required.

With some extra work, one can prove that $\frac{\sqrt{65}}{8}$ is the minimum possible diameter that will allow the three disks to cover the square.

5. Show that for any positive integer $n \geq 2$ we have

$$2^{2^n(n-2)+n+2} < (2^n)! < 2^{2^n(n-1)+1}.$$

(For a positive integer M, $M!$ stands for the product $1 \cdot 2 \cdot 3 \cdots (M-1) \cdot M$.)

SOLUTION. First we establish the following formulas:

$$1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1$$

and

$$1 \cdot 2^1 + 2 \cdot 2^2 + \cdots + n2^n = (n-1)2^{n+1} + 2.$$

To establish the first formula notice that if the sum in the left-hand side is S_n, then

$$2S_n = 2 + 4 + \cdots + 2^{n+1} = S_n - 1 + 2^{n+1},$$

which implies $S_n = 2^{n+1} - 1$.

If L_n is the sum in the left-hand side of the second formula, then

$$\begin{aligned} 2L_n &= 1 \cdot 2^2 + 2 \cdot 2^3 + \cdots + (n-1)2^n + n2^{n+1} \\ &= (1-1) \cdot 2^1 + (2-1) \cdot 2^2 + \cdots (n-1) + n2^{n+1} \\ &= L_n - (S_n - 1) + n2^{n+1} \\ &= L_n - (2^{n+1} - 2) + n2^{n+1} = L_n + (n-1)2^{n+1} + 2. \end{aligned}$$

which implies $L_n = (n-1)2^{n+1} + 2$.

Now let us get back to the statement of the problem. To obtain the first inequality let us split the factors in $(2^n)!$ into the following groups. The first group will contain just one number, 1. The second group will consist of two numbers, 2 and 3. The third group will have four numbers, $4, 5, 6$, and 7. And so on: the $(n-1)$-st group will consist of 2^{n-1} numbers. Notice that by the first formula we established above, our $n-1$ groups contain $2^n - 1$ factors from $(2^n)!$, that is, all but the last one 2^n. The same formula implies that k-th group starts with 2^{k-1}.

Notice that the numbers in each group are increasing. Therefore, all of them are greater or equal to the first number in the group. Hence, the product of all 2^{k-1} numbers in k-th group is at least $(2^{k-1})^{2^{k-1}} = 2^{(k-1)2^{k-1}}$. Combining the products in all groups, and multiplying by the last factor 2^n, we obtain

$$(2^n)! > 2^{0\cdot 2^0 + 1\cdot 2^1 + \cdots + (n-1)\cdot 2^{n-1} + n} = 2^{(n-2)2^n + 2 + n}.$$

The inequality is strict because in each group, each number other than the first one is strictly greater than the first number. (Because $n \geq 2$, we will have at least two groups.)

To prove the second inequality split the factors in the following groups. Discard the first number, 1, because it does not affect the product. (Alternatively, we can put it in the 'zero-th' group.) The first group contains one number, 2. The second consists of two numbers, 3 and 4. The $(n-1)$-st group contains the last 2^{n-1} numbers (now including 2^n). Notice that all the factors in the k-th group are at most the last number in that group, 2^k. It follows that

$$(2^n)! < 2^{1\cdot 2^0 + 2\cdot 2^1 + \cdots + 2^{n-1}\cdot n} =$$

$$= 2^{(1\cdot 2^1 + \cdots + (n-1)\cdot 2^{n-1}) + (1 + 2 + \cdots + 2^{n-1})} =$$

$$= 2^{((n-2)2^n + 2) + (2^n - 1)} = 2^{(n-2)2^n + 1}.$$

Again, the inequality is strict, because in each group, all numbers but the last one are strictly less than the last number.