WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

**PROBLEM SET IV (2023-2024)**   **January 2024**

1. A sequence of numbers is formed according to the following rules: in the first place, we put "1", and, if the number in the k-th place is N, then we put $2N + 1$ in the $(2k)$-th place and $2N$ in the $(2k + 1)$-st place. (So the first three numbers of the sequence are $1, 3, 2$.) Show that every positive integer will eventually appear in this sequence, but only once.

2. Is it true that for any $x \geq 1$ the following statement holds?
$$\lfloor \sqrt{x} \rfloor = \left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor$$
Recall the definition of $\lfloor y \rfloor$: for a real number y this is the *floor* of y, the greatest integer less than or equal to y.

3. Call a positive integer *nice* if all of its digits are from 2, 3, 4, 5, 6, 7. How many 100-digit nice numbers are multiples of 2^{100}?

4. Consider a unit cube with vertices $ABCDEFGH$, as shown in the picture. Identify the set of points on the face $ABCD$ which are of equal distance from the vertices A and G.
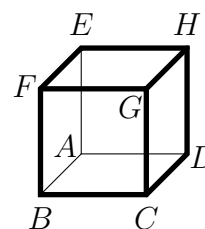
5. A warehouse stores 100 boxes containing nuts, bolts, and washers. Each box contains at least one nut, one bolt, and one washer. Prove that one can choose 51 boxes so that together they contain more than half of all the nuts, more than half of all the bolts, and more than half of all the washers.

You are invited to submit a solution even if you get just one problem. Please do not write your solutions on this problem page. Remember that solutions require a proof or justification.

Find old and current problems and information about the talent search at: `http://www.math.wisc.edu/talent`
Find an introduction to techniques for solving problems like these at `https://go.wisc.edu/551pe6`

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