WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET II (2023-2024)

1. The positive integers 1, 2, 3, through 200 are written down in a list. Alex draws circles around 10 of these numbers, and the mean of Alex's 10 numbers is 40. Beth draws circles around 20 of the numbers, and the mean of Beth's 20 numbers is 70. Show that the mean of the numbers with either one or two circles around them can be 80 but not 82.

SOLUTION. The numbers circled by Alex have sum $10 \cdot 40 = 400$, and the numbers circled by Beth have sum $20 \cdot 70 = 1400$. Suppose there are k numbers circled by both Alex and Beth, and the sum of those k numbers is s. For the circled numbers to have mean 80, it must be that

$$\frac{400 + 1400 - s}{30 - k} = 80, \qquad \text{and hence} \qquad k = 30 - \frac{1800 - s}{80}.$$

Since we cannot have more than 10 numbers circled twice, we must have $0 \le k \le 10$, which implies that

$$20 \le \frac{1800 - s}{80} \le 30,$$

with $1800 - s$ being an integer multiple of 80.

$k = 30 - \frac{1800-s}{80}$ is a integer between 0 and 10. Thus $1800 - s \le 1800$ is a multiple of 80 that is at least $80 \cdot 20 = 1600$. This means that $1800 - s$ can be $1600, 1680$, or 1760, which gives the following possibilities for (k, s): $(8, 40)$, $(9, 120)$, and $(10, 200)$.

In the case of $k = 10$ and $s = 200$, all of Alex's numbers are circled with two circles, so their mean would need to be $\frac{200}{10} = 20$, which is not possible because that mean is 40. In the case of $k = 9$ and $s = 120$, all but one of Alex's numbers are circled with two circles, so Alex's number with only one circle would need to be $400 - 120 = 280$, which is not possible because none of the integers in the list exceed 200. But $k = 8$ and $s = 40$ is possible. Indeed, Alex could have circled 1, 2, 3, 4, 5, 6, 7, 12, 160, 200 while Beth circled 1, 2, 3, 4, 5, 6, 7, 12, 101, 102, 103, 104, 105, 106, 120, 121, 122, 123, 124, 129.

For the mean of the circled numbers to have mean 82, $1800 - s$ must be a multiple of 82, between $82 \cdot 20 = 1640$ and 1800. The only such numbers are 1640 and 1722, which lead to the following possible (k, s) pairs: $(9, 78)$ and $(10, 160)$. In the case of $k = 10$ and $s = 160$, the mean of the Alex's numbers is 16, which is not possible because that mean is 40. In the case of $k = 9$ and $s = 78$, all but one of Alex's numbers are circled with two circles, so Alex's number with only one circle would need to be $400 - 78 = 322$, which is not possible because none of the integers in the list exceed 200.

2. There are 9 pieces of cheese on a plate (possibly with different weights). Is it always possible to cut exactly one of the pieces into two parts so that the resulting 10 pieces can be divided into two groups of 5 pieces each with the two groups having equal weight?

SOLUTION. Let the weights of pieces from least to greatest be

$$a_1 \le a_2 \le a_3 \le a_4 \le a_5 \le a_6 \le a_7 \le a_8 \le a_9,$$

and let $S = a_1 + a_2 + \cdots + a_9$ be the sum of all weights. Because

$$a_1 + a_3 + a_5 + a_7 \le a_2 + a_4 + a_6 + a_8 \le a_3 + a_5 + a_7 + a_9,$$

it follows that both $a_1 + a_3 + a_5 + a_7$ and $a_2 + a_4 + a_6 + a_8$ are less than $\frac{1}{2} \cdot S$. Indeed:

$$2(a_1 + a_3 + a_5 + a_7) \le a_1 + a_3 + a_5 + a_7 + a_2 + a_4 + a_6 + a_8 < S$$

and

$$2(a_2 + a_4 + a_6 + a_8) \le a_2 + a_4 + a_6 + a_8 + a_3 + a_5 + a_7 + a_9 < S.$$

Now cut the largest piece with weight a_9 into two pieces with weights $x = \frac{1}{2} \cdot S - (a_1 + a_3 + a_5 + a_7)$ and $y = \frac{1}{2} \cdot S - (a_2 + a_4 + a_6 + a_8)$. These numbers are both positive, and their sum is a_9:

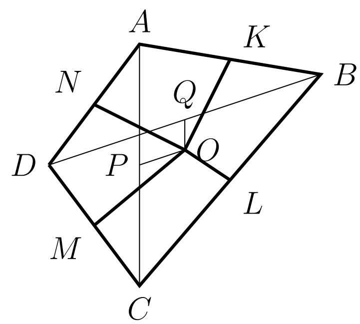
$$x + y = S - (a_1 + a_3 + a_5 + a_7 + a_2 + a_4 + a_6 + a_8) = a_9.$$

Now we have

$$a_1 + a_3 + a_5 + a_7 + x = \frac{1}{2}S \quad \text{and} \quad a_2 + a_4 + a_6 + a_8 + y = \frac{1}{2}S,$$

so the two groups a_1, a_3, a_5, a_7, x and a_2, a_4, a_6, a_8, y satisfy the required conditions.

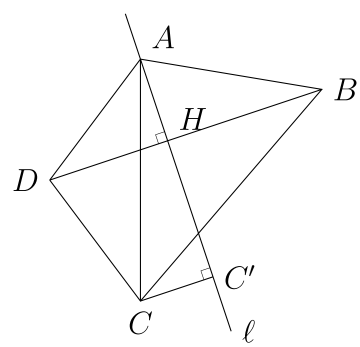
3. $ABCD$ is a convex quadrilateral. Denote by P and Q the midpoints of its diagonals. Draw through P and through Q lines parallel to the other diagonal, and let O be their intersection point. Let K, L, M, and N be the midpoints of sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, respectively, and draw segments $\overline{KO}$, $\overline{LO}$, $\overline{MO}$, and $\overline{NO}$, as shown. This subdivides $ABCD$ into four smaller quadrilaterals: $ONAK$, $OKBL$, $OLCM$, and $OMDN$. Prove that these four quadrilaterals all have the same area.



SOLUTION. We will begin by proving a general formula for the area of a quadrilateral: it equals one half of the product of a diagonal and the projection of the other diagonal onto the line perpendicular to the first diagonal. Namely, in a quadrilateral $ABCD$, let ℓ be the line running through A perpendicular to line BD and let C' be the base of the perpendicular dropped onto ℓ from C (so $\ell \perp BD$ and $CC' \perp \ell$). Let Area(X) be the area of polynomial X, and let Dist(Y, Z) be the distance from a point Y to a line Z. Then

$$\text{Area}(ABCD) = \frac{|BD| \cdot |AC'|}{2}.$$

Proof of the formula. The diagonal $\overline{BD}$ divides the quadrilateral $ABCD$ into two triangles, so



$$\text{Area}(ABCD) = \text{Area}(\triangle ABD) + \text{Area}(\triangle BCD).$$

Next,

$$\text{Area}(\triangle ABD) = \frac{|BD| \cdot \text{Dist}(A, BD)}{2}$$

and

$$\text{Area}(\triangle BCD) = \frac{|BD| \cdot \text{Dist}(C, BD)}{2}.$$

Let H be the intersection of ℓ and line BD. Then

$$\text{Dist}(A, BD) = |AH|,$$

and, as $CC' \parallel BD$,

$$\text{Dist}(C, BD) = \text{Dist}(C', BD) = |CH|.$$

Hence,

$$\text{Area}(ABCD) = \frac{|BD| \cdot |AH|}{2} + \frac{|BD| \cdot |CH|}{2} = \frac{|BD| \cdot (|AH| + |CH|)}{2} = \frac{|BD| \cdot |AC'|}{2},$$

as claimed.

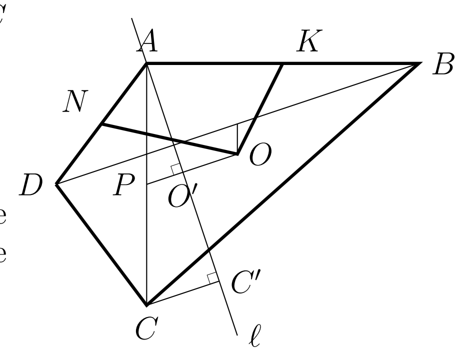
Remark. In this argument, we assumed that the quadrilateral $ABCD$ is convex, so that A and C are on the opposite sides of the line BD, as this is the only case that we need. However, the formula holds in full generality.

We now return to the problem. Let us prove that the area of each of the four smaller quadrilaterals equals $\frac{1}{4} \cdot \text{Area}(ABCD)$. Because the situation is completely symmetric, it suffices to consider one of them; let us show that

$$\text{Area}(ONAK) = \frac{\text{Area}(ABCD)}{4}.$$

As above, let us draw a line ℓ through A perpendicular to line BD, and let C' be the base of the perpendicular dropped from C onto BD. Then

$$\text{Area}(ABCD) = \frac{|BD| \cdot |AC'|}{2}.$$



Notice now that BD and NK are parallel, because NK is the midline of $\triangle ABD$. Hence, ℓ is also perpendicular to NK. Thus, if O' is the base of the perpendicular dropped onto ℓ from O,

$$\text{Area}(ONAK) = \frac{|NK| \cdot |AO'|}{2}.$$

Since NK is the midline of $\triangle ABD$,

$$|NK| = \frac{|BD|}{2}.$$

Finally, we know that $OP || BD$, and, therefore, $OP \perp \ell$, and because $OO' \perp \ell$, we see that $PO' \perp \ell$. (We assume here that P is the midpoint of the diagonal $\overline{AC}$, as in the diagram.) This implies that $\triangle ACC'$ and $\triangle APO'$ are similar because they are right triangles sharing a common angle at A. By construction

$$|AP| = \frac{|AC|}{2},$$

and, therefore,

$$|AO'| = \frac{|AC'|}{2},$$

which gives

$$\text{Area}(ONAK) = \frac{1}{2} \cdot |NK| \cdot |AO'| = \frac{1}{2} \cdot \frac{|BD|}{2} \cdot \frac{|AC'|}{2} = \frac{1}{4}\text{Area}(ABCD),$$

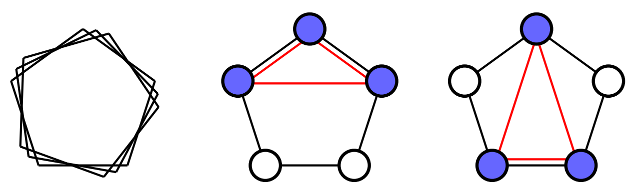
as required.

4. A regular polygon with 5055 sides has 2023 of its vertices painted blue. Show that there is an isosceles triangle whose vertices are three of these blue vertices.

SOLUTION. First notice that the 5055 vertices of the regular polygon are the vertices of 1011 regular pentagons with disjoint sets of vertices. Indeed, choose any vertex A_1 of the regular polygon, then choose vertex A_2 1011 vertices clockwise from A_1, then A_3 1011 vertices clockwise from A_2, and so on to choose A_1, A_2, A_3, A_4, and A_5. These are the vertices of a regular pentagon. After that, for the next pentagon, choose the first vertex clockwise from A_1, and repeat the process.

After choosing 1011 pentagons as above, notice that at least one of the pentagons will have 3 blue vertices. Indeed, if each pentagon had 2 or fewer blue vertices, then there would be at most $2 \cdot 1011 = 2022$ blue vertices.

It is left to notice that any 3 vertices of a regular pentagon are the vertices of an isosceles triangle. Indeed: if the three chosen vertices are next two each other, then we get two sides in the triangle that are equal to the sides of the regular pentagon, otherwise we get two sides in the triangle that are equal to one of the diagonals of the regular pentagon. (See the diagrams below.)



5. Starting with a positive integer, if it is even, divide it by two, but if it is odd, add 7 to it. Call a positive integer *awesome* if applying this step a finite number of times can produce the number 1. For example, 9 is awesome because the steps produce 9, 16, 8, 4, 2, 1. Find the number of awesome positive integers between 1 and 2023.

SOLUTION. We will show that a positive integer is awesome exactly if its remainder when divided by 7 is equal to 1, 2, or 4.

Suppose that an integer n has remainder a when divided by 7. This means that n can be written in the form of $7k+a$ for some integer k. If n is odd, then the next step produces $n+7 = 7(k+1)+a$, which has the same remainder when dividing by 7 as n has. If n is even, then the next step produces $2n = 2(7k + a) = 14k + 2a$, and its remainder when divided by 7 is the same as the remainder when $2a$ is divided by 7. This list shows how the remainders change after doubling:

$$0 \mapsto 0, \quad 1 \mapsto 2, \quad 2 \mapsto 4, \quad 3 \mapsto 6, \quad 4 \mapsto 1, \quad 5 \mapsto 3, \quad 6 \mapsto 5.$$

Each of the possible seven remainders can be obtained from exactly one possible remainder as the result of doubling, which means that if n is even, then the remainder of $n/2$ when divided by 7 is determined by the remainder when n is divided by 7 using the following rules (just reversed the arrows in the previous list):

$$0 \mapsto 0, \quad 1 \mapsto 4, \quad 2 \mapsto 1, \quad 3 \mapsto 5, \quad 4 \mapsto 2, \quad 5 \mapsto 6, \quad 6 \mapsto 3.$$

Note that if n has a remainder of 0, 3, 5, or 6 when divided by 7, then performing one step of the procedure will give an integer that also has a remainder of 0, 3, 5, or 6 when divided by 7. Thus, no integer of this form can ever produce a remainder of 1, so no number with remainder 0, 3, 5, or 6 when divided by 7 can be awesome.

Now suppose that a positive integer has remainder 1, 2, or 4 modulo 7. Then in each step its remainder when divided by 7 will be from this list (when we add 7 the remainder does not change, when we divide by two, they change according to $1 \mapsto 4 \mapsto 2 \mapsto 1$). If an integer is greater than 7, it decreases when performing one or two steps on it. Indeed, if it is even, then it becomes less after one step. If it is odd, then it will become less after two steps, because when $x > 7$

$$\frac{x+7}{2} < x.$$

This means that any positive integer can be reduced to a positive integer that is at most 7 after a finite number of steps, as it will always decrease by at least 1 if it is greater than 7. Thus, if the starting number has remainder 1, 2, or 4 when divided by 7, after a finite number of steps, the procedure will produce either 1, 2, or 4, and each of these numbers are awesome, so the starting number is awesome.

This shows that the awesome positive integers are indeed the ones with remainder 1, 2, or 4 modulo 7. Since $2023/7 = 289$, there are exactly $3 \cdot 289 = 867$ such integers between 1 and 2023.