WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET I (2023-2024)

1. Each of the 2023 guests sitting around a large round table either always tells the truth or always lies. Each guests says "there is exactly one truth-teller sitting next to me." Show that all the guests are liars.

SOLUTION. First note that if all guests are liars, then the statement "there is exactly one truth teller sitting next to me" is false for each guest, and since all of them are liars, they can all make this statement.

Suppose the statement is false, and not all guests are liars. There is at least one truthteller; let us choose him as guest one. As he is telling the truth, one of his neighbors must also be a truthteller. Let us call him guest two, and continue numbering guests moving around the circle (either clockwise or counterclockwise, depending on whether guest two is on guest one's right side or on his left).

We can now determine honesty of all the guests one by one. Namely, guest two is telling the truth, and we know that one of his neighbors (guest one) is a truthteller, therefore, the other (guest three) must be a liar. Now, guest three already has a truth-telling neighbor (guest two), which means his other neighbor cannot possibly be a liar, otherwise guest three would be telling the truth (which is against his nature). Thus, guest four is a truthteller as well.

We now see the repeating pattern of $TTLTTLTTL\ldots$ (where T is a truthteller and L is a liar), with every third guest a liar. However, as the total number of guests (2023) is not divisible by 3, the pattern will be broken as we complete the full circle, leading to a contradiction.

ALTERNATE SOLUTION. As in the previous solution, suppose that not all guests are liars. We claim that in this case, every liar is sitting between two truthtellers. Indeed, the only other option is for a liar to be between two liars (if a liar sits between a liar and a truthteller, his statement is true, which is not allowed). But then, each of his neighbors must be between two liars as well, and so on, showing that all the guests are liars. However, we have excluded this possibility.

Now, we know that each truthteller sits between a truthteller and a liar, while every liar sits between two truthtellers. Suppose that there are n truthtellers. If now each truthteller writes down the name of the liar sitting next to them, then we get a list with n names. On the list, each liar's name will show up exactly twice (once for each of his two truth-telling neighbors). Hence, we must have $n/2$ liars. But we must have $n + n/2 = 2023$ (the total number of guests), which would lead to $n = \frac{4046}{3}$, a non-integer number. This means that our assumption was incorrect; the only way the situation that the problem describes can happen is if all the guests are liars.

2. Pat and Harper each bicycle uphill from Aston to Benfield and then downhill from Benfield to Coville. Each rider has a constant uphill speed and a constant downhill speed, but Pat climbs hills faster than Harper, and Harper rides downhill faster than Pat. Harper begins the ride 22 minutes before Pat begins, but Pat passes Harper when they are half way between Aston and Benfield. Then Harper passes Pat when they are half way between Benfield and Coville. Which bicyclist completes the ride from Aston to Coville in the least amount of time?

SOLUTION. Pat and Harper finish the trip in the same amount of time. Indeed, Pat and Harper are at the same locations at the same time when they are half way between Aston and Benfield, and again when they are half way between Benfield and Coville. That means that they each travel half the distance from Aston to Benfield plus half the distance from Benfield to Coville in the same amount of time. Thus, they travel the entire distance from Aston to Benfield plus the entire distance from Benfield to Coville in twice that amount of time, which is the same for both riders.

3. I wrote down the positive integer number a twice next to each other to get a new number b. (E.g. if $a = 123$, then b would be 123123.) What are the possible values of the fraction $\frac{b}{a^2}$ if we know that the fraction is an integer?

SOLUTION. Suppose that a has n digits. Then the number b is equal to the sum of a and the number a followed by n zeros (which is $a \cdot 10^n$). Hence,

$$b = a \cdot 10^n + a = a(10^n + 1).$$

This means that the fraction $\frac{b}{a^2}$ is equal to $\frac{10^n+1}{a}$. Denote the integer $\frac{b}{a^2}$ by c.

When $n = 1$, then $10^n + 1 = 11$. The only single digit divisor of 11 is 1. This gives $a = 1$, $b = 11$, and $c = \frac{b}{a^2} = 11$ as a possible solution.

The least n-digit number is 10^{n-1}. Hence, $a \geq 10^{n-1}$. Assume $n \geq 2$. Then

$$10^n + 1 < 10^n + 10^{n-1} = 11 \cdot 10^{n-1} \leq 11a.$$

Hence, $c = \frac{10^n+1}{a} < 11$, and since c is an integer, $c = \frac{10^n+1}{a} \leq 10$. Note that c cannot be equal to 1, because this would lead to $a = 10^n + 1$ which has $n + 1$ digits, not n. Hence, $2 \leq c \leq 10$.

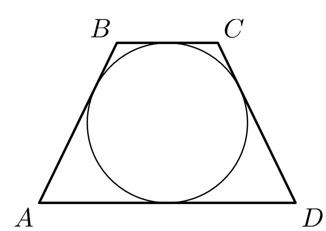
c is a divisor of $10^n + 1$, an odd number, so it has to be odd. The number $10^n + 1$ cannot be divisible by 5 because 10^n is a multiple of 5. It also cannot be divisible by 3 or 9 because $10^n - 1$ is divisible by both (since it is the number made up from n 9s), and $10^n + 1 = 2 + (10^n - 1)$ would give a remainder of 2 when divided by 3 or 9.

The only possible odd integer value that is left between 2 and 10 is 7. This is a possible answer: $1001 = 10^3 + 1$ is divisible by 7, and $\frac{1001}{7} = 143$ gives a possible value for c: $\frac{143143}{143^2} = 7$.

Therefore, the possible values for $\frac{b}{a^2}$ are 7 and 11.

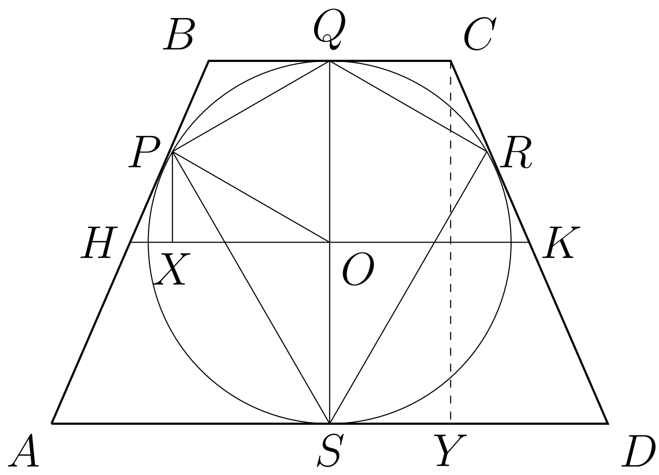
Remark. It can be shown that $10^n + 1$ is divisible by 7 exactly when $n = 6k + 3$. This means that the possible values for a are 1 and the numbers $\frac{10^{6k+3}+1}{7}$ for $k = 1, 2, \ldots$.

4. The four sides of an isosceles trapezoid of area 5 are each tangent to a circle of radius 1. Find the area of the quadrilateral whose vertices are the points of tangency.



SOLUTION. Denote the trapezoid $ABCD$ with $|AB| = |CD|$ and $BC \parallel AD$, as on the picture.

Let the points of tangency of the sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ be P, Q, R, and S, respectively. We need to find the area of the quadrilateral $PQRS$. It is composed of two triangles $\triangle PQS$ and $\triangle QRS$. Since the trapezoid is isosceles, the entire picture is symmetric around the line QS. In particular, the two triangles have equal area, and hence, the area of $PQRS$ equals twice the area of $\triangle PQS$. We also see that $\overline{QS}$ is the diameter of the circle, and therefore, $QS = 2$.



Let H be the midpoint of $\overline{AB}$ and K be the midpoint of $\overline{CD}$. Thus, $\overline{HK}$ is the midsegment of the trapezoid $ABCD$, and therefore,

$$HK = \frac{BC + AD}{2}.$$

The distance between $\overline{BC}$ and $\overline{AD}$ is equal to 2, the diameter of the circle. The area of the trapezoid equals the product of the height and $\frac{BC+AD}{2} = HK$. We are told that the area equals 5, and therefore, $HK = \frac{5}{2}$.

Let O be the center of the circle. It is also the midpoint of $\overline{HK}$, therefore, $HO = \frac{5}{4}$. Let X be its base of the perpendicular from the point P to the midsegment $\overline{HK}$. The length XO is equal to the distance from P to the line QS, and therefore, the area of $\triangle PQS$ equals $\frac{1}{2} \cdot XO \cdot QS = \frac{1}{2} \cdot XO \times 2 = XO$. As we already know, the area of the quadrilateral $PQRS$ is twice this amount, so it equals $2 \cdot XO$.

It remains to find the length XO. Consider the triangles $\triangle HPO$ and $\triangle PXO$. They are both right triangles: $\angle HPO = 90^\circ$ because $\overline{HP}$ is tangent to the circle and $\overline{PO}$ is a radius at the point of tangency, while $\angle PXO = 90^\circ$ because $\overline{PX}$ is a perpendicular dropped onto $\overline{HK}$. The two triangles also share the angle at the vertex O. Therefore, they are similar, and we see that

$$\frac{HO}{PO} = \frac{PO}{XO}.$$

We already know that $HO = \frac{5}{4}$; also $PO = 1$ because it is a radius of the circle. From this, we see that

$$XO = \frac{PO^2}{HO} = \frac{4}{5}.$$

Finally, the area of $PQRS$ equals $2 \cdot \frac{4}{5} = \frac{8}{5}$.

ALTERNATE SOLUTION. Let $a = CQ = CR = \frac{1}{2} \cdot BC$ and $b = DS = DR = \frac{1}{2} \cdot AD$. Let Y be the projection of C onto $\overline{AD}$. Then $CD = a + b$ and $DY = b - a$. The area of $ABCD$ is $2(a + b) = 5$, so $a + b = \frac{5}{2}$. Applying the Pythagorean Theorem to $\triangle CYD$ gives

$$CD^2 = CY^2 + DY^2 = 2^2 + (b - a)^2 = (a + b)^2 = \frac{25}{4},$$

from which $b - a = \frac{3}{2}$. Thus, $a = \frac{1}{2}$ and $b = 2$. Because $\frac{DR}{CR} = \frac{b}{a} = 4$, it follows that $PR = \frac{4}{5} \cdot BC + \frac{1}{5} \cdot AD = \frac{8}{5}$. The area of kite $PQRS$ is then $\frac{1}{2} \cdot PR \cdot QS = \frac{8}{5}$, as above.

5. We say that a positive integer *contains 23* if among its digits, we can find 2 and 3 next to each other in that order. Similarly, we say that a positive integer *contains 22* if there are two 2s next to each other among its digits. (E.g. the number 12342 contains 23 but not 22.) Show that there are more 2023-digit integers that contain 23 than those that contain 22.

SOLUTION. For $n > 1$, consider the n-digit positive integers and let

- a_n be the number of such integers that contain neither 22 nor 23 and do not end in a 2,
- b_n be the number of such integers that contain 22 but not 23 and do not end in a 2,
- c_n be the number of such integers that contain 23 but not 22 and do not end in a 2,
- d_n be the number of such integers that contain both 22 and 23 and do not end in a 2,
- e_n be the number of such integers that contain neither 22 nor 23 and end in a 2,
- f_n be the number of such integers that contain 22 but not 23 and end in a 2,
- g_n be the number of such integers that contain 23 but not 22 and end in a 2,
- h_n be the number of such integers that contain both 22 and 23 and end in a 2.

Let R_n be the number of n-digit positive integers that contain 23 minus the number of n-digit positive integers that contain 22, that is, $R_n = (c_n + g_n) - (b_n + f_n)$. We will show by mathematical induction that $c_n > b_n$ and $R_n > 0$ for all $n > 2$. It is easy to see that $b_3 = 8$, $c_3 = 17$, $f_3 = 9$, and $g_3 = 1$, so $c_3 - b_3 = 9 > 0$ and $R_3 = 1 > 0$. Assume that for some $k \geq 3$ that $c_k > b_k$ and $R_k > 0$. Considering what happens when a k-digit positive integer is turned into a $(k+1)$-digit positive integer by adding one more digit to the end shows that

$$\begin{aligned} b_{k+1} &= 9b_k + 8f_k, \\ c_{k+1} &= 9c_k + 9g_k + e_k, \\ f_{k+1} &= b_k + f_k + e_k, \text{ and} \\ g_{k+1} &= c_k. \end{aligned}$$

Thus, $c_{k+1} - b_{k+1} = 8\big((c_k + g_k) - (b_k + f_k)\big) + (c_k - b_k) + e_k = 8R_k + (c_k - b_k) + e_k > 0$, and

$$\begin{aligned} R_{k+1} &= (c_{k+1} + d_{k+1} + g_{k+1} + h_{k+1}) - (b_{k+1} + d_{k+1} + f_{k+1} + h_{k+1}) \\ &= (c_{k+1} + g_{k+1}) - (b_{k+1} + f_{k+1}) \\ &= \big((9c_k + 9g_k + e_k) + c_k\big) - \big((9b_k + 8f_k) + (b_k + f_k + e_k)\big) \\ &= 9(c_k + g_k - b_k - f_k) + (c_k - b_k) \\ &= 9R_k + (c_k - b_k) > 0. \end{aligned}$$

This completes the proof by mathematical induction.

ALTERNATE SOLUTION. We will show that if $n \geq 3$, then there are more n-digit positive integers containing 23 than ones containing 22. Fix $n \geq 3$ and let X_{22} be the collection of n-digit numbers that contain 22, but not 23, and X_{23} be the collection of n-digit numbers that contain 23, but not 22. It is sufficient to show that X_{23} has more elements than X_{22}.

For a positive integer number a *block of twos* is a sequence of consecutive digits of twos that cannot be made into a longer sequence by including an additional digit before or after the sequence. For example, there are three blocks of twos in the number 12223123122: 12223123122.

An integer is an element of X_{22} if it has n digits and the following statements are true:

- it has at least one block of twos of length at least 2 (so that it contains 22),
- each block of twos within the number is followed by a digit not equal to 3, or the block ends with the last digit of the number (so that it does not contain 23).

Now consider the following transformation on such a number: for each block of twos with length at least 2, replace the block with a block of the same size that begins with 2 and alternates between 2 and 3: $2323\ldots$. The resulting integer will contain 23 (since we updated at least one block of twos), but not 22 (since we transformed all blocks of two of length at least 2, and we could not create a new block of two of length at least 2.) Moreover, our transformation will produce different outputs for different input numbers. Indeed, if a number is the result of the transformation, then we can identify the original blocks of twos of length at least 2 by looking for the maximal blocks of the form $23\ldots23$ or $23\ldots2$.

On the other hand, there is at least one n-digit integer that contains 23 but not 22 that cannot be obtained as the result of our transformation: $9999\ldots9233$ (for $n = 3$ the number is 233). If this number is the result of our transformation, then the original number would end in 223, which would make that number contain both 22 and 23.

To summarize: for $n \geq 3$ we managed to transform each element of the collection X_{22} into an element of X_{23}, in a way that different numbers are transformed into different ones, and there is at least one element of X_{23} that is not obtained as the result of the transformation. This means that X_{23} has more elements than X_{22}.