WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET V (2022-2023)

1. An empty meeting room has a large round table with 60 chairs around it. A class of 30 students comes into the room, and the students sit down on the chairs around the table (leaving 30 of the chairs empty). Show that one can always find six chairs next to each other so that there are exactly three of them are taken by the students and three of them are empty.

SOLUTION. Let us call a block of six chairs next to each other *balanced* if there are exactly three chairs taken within the block, *heavy*, if there are four or more chairs taken, and *light*, if there are two or fewer chairs are taken.

Divide the 60 chairs into 10 blocks of six chairs next to each other. If there is a balanced block among these ten, then we are done, so let us assume that this is not the case. If all ten blocks are heavy, then we would have at least $4 \times 10 = 40$ chairs taken among them, and since these blocks do not overlap, we would get a contradiction. (Since there are exactly 30 chairs that are taken.) Similarly, if all ten blocks are light, then the number of empty chairs would be at least 40, which is again a contradiction. This means that we must have at least one heavy and at least one light block among these 10 blocks, and, moreover, we must have a heavy and a light block next to each other. (Otherwise all 10 blocks would be light or heavy.)

Consider these two blocks next to each other, and label the chairs within them $1, 2, \ldots, 12$ going clockwise. That means that one of the blocks contains the chairs 1 through 6, and the other contains the chairs 7 through 12. Consider also the blocks corresponding to the chairs numbered 2-7, 3-8, 4-9, 5-10, and 6-11. Note that the number of students in the blocks $k, \ldots, k+5$ and $k+1, \ldots, k+6$ are either equal to each other (if chairs k and $k+6$ are both empty or both taken), or differ exactly by one (if exactly one of the chairs k and $k+6$ is taken, and the other is empty). That means that if we consider the blocks $k, \ldots, k+5$ and $k+1, \ldots, k+6$ for some $1 \leq k \leq 6$, then we cannot have one of them heavy and one of them light (since then the difference of the number of students would be at least 2). That means that the blocks corresponding to 2-7, 3-8, 4-9, 5-10, 6-11 must contain a balanced block: otherwise all the blocks $1-6, \ldots, 7-12$ are heavy or light, and since the first and last one are different (because of our assumption), we would have a heavy and a light block next to each other in this list which is impossible.

2. Is it possible to color each integer greater than 1 with either red, green, or blue so that all three colors are used, and for every two integers x and y colored in two different colors, the integer $x \cdot y$ is colored with the third color?

SOLUTION. The answer is no. Let the three colors be red, green, and blue. Without loss of generality, let 2 be colored red, and 4 be colored red or green. Let number k be colored blue. If the property holds, $2k = 2 \cdot k$ would be colored green, $4k = 4 \cdot k$ would not be colored the color of k, so it would not be blue. But $4k = 2 \cdot 2k$, which must be colored blue. This is a contradiction.

3. Each of the following expressions contain 2023 fractions. Which expression has the greater value?

$$\cfrac{2}{2+\cfrac{2}{2+\cfrac{2}{\ddots+\cfrac{2}{2+\cfrac{2}{2}}}}} \quad \text{or} \quad \cfrac{3}{3+\cfrac{3}{3+\cfrac{3}{\ddots+\cfrac{3}{3+\cfrac{3}{3}}}}}$$

SOLUTION. First note that $2+\frac{2}{2}=3<4=3+\frac{3}{3}$. Then, if $0<A<B$, it follows that $\frac{1}{A}>\frac{1}{B}$, so $1+\frac{1}{A}>1+\frac{1}{B}$ and

$$\frac{2}{2+\dfrac{2}{A}}=\frac{1}{1+\dfrac{1}{A}}<\frac{1}{1+\dfrac{1}{B}}=\frac{3}{3+\dfrac{3}{B}}.$$

Thus, if $A<B$, where A and B are the expressions each containing k fractions using 2s and 3s, respectively, then the inequality is maintained for the expressions each containing $k+2$ fractions. It follows that of the expressions with 2023 fractions, the one with the 2s is less than the one with the 3s.

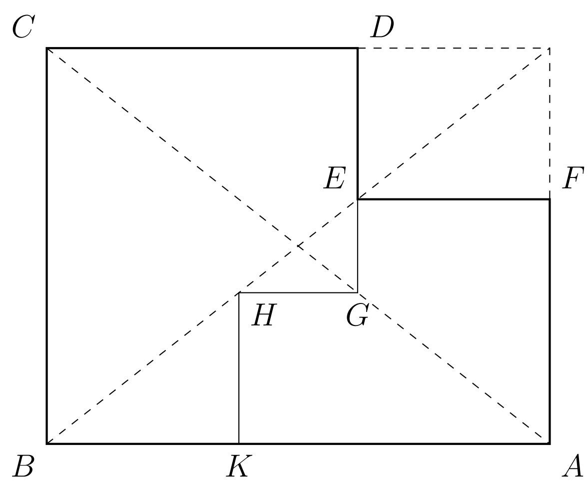
4. Give an example of a hexagon satisfying the following two properties: (a) all the interior angles of the hexagon are either 90 or 270 degrees, and (b) the hexagon can be cut into two unequal hexagons geometrically similar to the original hexagon.

SOLUTION.

Let α be the positive root of the equation $x^4-x^2-1=0$ ($\alpha=\sqrt{\frac{1+\sqrt{5}}{2}}\approx 1.272$). Then the hexagon $ABCDEF$, in which

$$AB:BC=BC:CD=CD:AF=AF:FE=FE:ED=\alpha,$$

can be cut as in the picture below. One can show that the resulting hexagons are similar to the original with coefficients $\frac{1}{\alpha}$ and $\frac{1}{\alpha^2}$.



To see this we can set $AB=1$, which means that the sides of the hexagon are

$$AB=1,\quad BC=\alpha^{-1},\quad CD=\alpha^{-2},\quad DE=\alpha^{-5},\quad FE=\alpha^{-4},\quad AF=\alpha^{-3}.$$

Now choose G on the extension of DE, so that $EG=\alpha^{-7}$, then choose H as in the picture so that $HG=\alpha^{-6}$ and HG is perpendicular to DG. Then we have

$$HK=AF-EG=\alpha^{-3}-\alpha^{-7}=\alpha^{-5},\qquad BK=CD-HG=\alpha^{-2}-\alpha^{-6}=\alpha^{-4},$$
$$DG=ED+EG=\alpha^{-5}+\alpha^{-7}=\alpha^{-3},\qquad AK=AB-BK=1-\alpha^{-4}=\alpha^{-2},$$

since $\alpha^4=1+\alpha^2$.

We can now check that the hexagon $BCDGHK$ has sides of length $\alpha^{-1}, \alpha^{-2}, \alpha^{-3}, \alpha^{-6}, \alpha^{-5}, \alpha^{-4}$ in clockwise order starting at BC, these are just α^{-1} times the side lengths of the hexagon $ABCDEF$. Since the respective angles agree, $BCDGHK$ is similar to $ABCDEF$.

The hexagon $KAFEGH$ has side lengths $\alpha^{-2}, \alpha^{-3}, \alpha^{-4}, \alpha^{-7}, \alpha^{-6}, \alpha^{-5}$ in counterclockwise order starting at AK. These are just α^{-2} times the corresponding side lengths of $ABCDEF$, and since the respective angles agree, we get that $KAFEGH$ is also similar to the hexagon $ABCDEF$.

5. A railroad line has 26 stations: $A, B, \ldots, Z$, in this order. A traveler wants to visit each station exactly once, starting and ending at A. He has a lot of time, so he goes from A to Z, then back to B, then to Y, then to C, and so on, until he finally goes from M to N and returns to A. Show that the length of this route is the maximum possible! Note however, that this is not the only route of maximum length; for instance, the traveler could have traced the same itinerary in reverse. How many routes of maximum length are there?

SOLUTION.

A) Consider 25 intervals: $AB, BC, CD, \ldots, YZ$, and let us count how many times each of the intervals gets traversed. On the way from A to Z, the traveler goes over each interval once, then on the way back he travels over each interval except AB again, and so on. After the traveler goes from M to N, the number of trips over each interval are given in the following table:

itinerary	AB	BC	CD	DE	...	LM	MN	NK	...	WX	XY	YZ
$A \to Z$	1	1	1	1	...	1	1	1	...	1	1	1
$A \to Z \to B$	1	2	2	2	...	2	2	2	...	2	2	2
$A \to Z \to B \to Y$	1	3	3	3	...	3	3	3	...	3	3	2
...							...					
$A \to \cdots \to M \to N$	1	3	5	7	...	23	25	24	...	6	4	2
$A \to \cdots \to M \to N \to A$	2	4	6	8	...	24	26	24	...	6	4	2

Let us prove that for each of the intervals, the total number of traverses is maximum; clearly, this implies that the total length of the route is maximum as well.

Indeed, for the interval AB, there is only one station (A) to its left, so it can be traversed at most twice – on the way to and from this station. Similarly, there is only one station (Z) to the right of YZ, so it can be traversed no more than twice – on the way to and from Z. The interval BC can be traversed at most four times: on the way to and from stations A and B, the same applies to XY: on the way to and from Y and Z. And so on, finally, the middle interval MN can be traversed no more than 26 times - during each of the 26 legs of the traveler's journey.

B) As explained in part A), the trip length will be maximum if the total number of times each of the intervals is traveled matches the last row of the table. In particular, the interval MN must be traversed 26 times, on each leg of the traveler's journey. This means that each time, the traveler needs to go from the station on the right half of the railroad $(A, \ldots, M)$ to the station on the left half $(N, \ldots, Z)$.

Conversely, we can see that this condition is sufficient for the distance to be maximum because it guarantees that each interval (not just MN) is traversed the maximum possible number of times.

Indeed, the traveler both goes from A to a station in the right half of the railroad, and returns to it from a station in the right half, meaning that AB gets traversed twice. BC will be traversed four times; the traveler goes from both A and B to somewhere in the right half of the railroad, and also goes to these stations from somewhere in the right half, and so on. For the intervals in the right half of the road, the situation is symmetric.

We now see that the itinerary must be of the form

$$A \to * \to ? \to * \to ? \to \cdots \to ? \to * \to A,$$

where $*$ (resp. ?) stands for a station on the right (resp. left) half of the board. We thus need to place 12 stations $B, \ldots, M$ (the stations on the left, excluding A) in 12 spots marked ?, and to place 13 stations $N, \ldots, Z$ (the stations on the right) in the 13 spots marked $*$. There are exactly

$$12! \times 13! = 2{,}982{,}752{,}926{,}433{,}280{,}000$$

ways to do so.