WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET IV (2022-2023)

1. Find all positive three-digit integers n such that if s is the sum of its digits, all digits of the number $n+s$ are equal, and all digits of $n-s$ are also equal (but possibly distinct from the digits of $n+s$).

SOLUTION. Suppose that the integer n satisfies the conditions of the problem, and let A and B be the numbers we get when we increase and decrease n by the sum of its digits, respectively. The sum of the digits of a three-digit integer is at most $3 \cdot 9 = 27$. Hence, the difference of A and B is at most $2 \cdot 27 = 54$. The three-digit integers that have the same three digits are all multiples of 111. Any two of these will have a difference that is a multiple of 111; hence, their difference is at least 111. This means that A and B cannot both be three-digit integers; one of them has either two digits or four digits. Because the difference between a two digit and a four digit number is greater than 54, the only possibilities are the following:

- A has four equal digits, and B has three equal digits, or
- A has three equal digits, and B has two equal digits.

If A has four digits, then it is at least 1111, and the difference between the three-digit integer B and A would be at least $1111 - 999 = 112 > 54$ which is impossible. Hence, A has three digits, and B has two digits. In that case A can only be 111 because, if it is greater than that, then $A \geq 222$, and the difference between A and B would be at least $222 - 99 = 123 > 54$. Hence, $A = 111$, which also implies that the first digit of n has to be equal to 1, and the second digit is either 0 or 1 (since $n < A$), and, hence, the sum of the digits of n is at most $1 + 1 + 9 = 11$. That implies that B is at least $100 - 11 = 89$, and since B has two equal digits, we must have $B = 99$. Thus, the only possible values are $A = 111$ and $B = 99$, which would mean that n must be the average of these: $n = \frac{1}{2}(99 + 111) = 105$. It turns out that this choice of n satisfies the conditions: the sum of its digits is 6, and $105 + 6 = 111$ and $105 - 6 = 99$. Hence, the only n that has the required property is $n = 106$.
Note: With a couple of extra steps one can show that the condition requiring n to have three digits can be removed.

2. A collection of 2023 real numbers has the property that its mean, median, and range all equal 2023. What is the greatest possible value of its maximum?

(If the numbers are, in order, $a_1 \geq a_2 \geq \cdots \geq a_{2023}$, the *mean* is $\frac{1}{2023}(a_1 + \cdots + a_{2023})$, the *median* is a_{1012}, and the *range* is $a_1 - a_{2023}$.)

SOLUTION. Let the numbers be $a_1 \geq \cdots \geq a_{2023}$. We are looking for the maximum possible value of a_1, given that $a_{2023} = a_1 - 2023$, $a_{1012} = 2023$, and

$$\frac{a_1 + \cdots + a_{2023}}{2023} = 2023.$$

From here we see that

$$a_1 + \cdots + a_{2023} = 2023^2.$$

Since the numbers are ordered, $a_2, \ldots, a_{1012}$ are all greater than or equal to $a_{1012} = 2023$, while $a_{1013}, \ldots, a_{2023}$ are all greater than or equal to $a_{2023} = a_1 - 2023$, and, therefore,

$$2023^2 = a_1 + \cdots + a_{2023} \geq a_1 + 1011 \times 2023 + 1011(a_1 - 2023),$$

giving the inequality $1012a_1 \leq 2023^2$, and hence

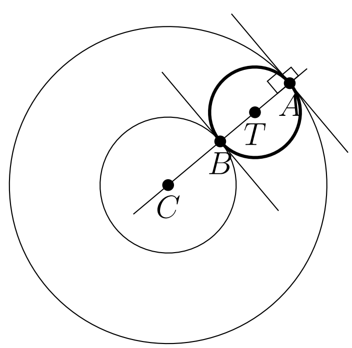
$$a_1 \leq \frac{2023^2}{1012} \approx 4044.001.$$

We also see that we can achieve this upper bound for a_1 by making sure all the inequalities we use are equalities, giving the numbers

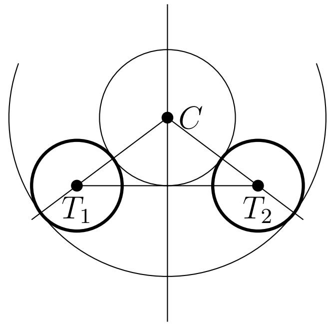
$$\frac{2023^2}{1012}, \underbrace{2023, \ldots, 2023}_{\text{1011 times}}, \underbrace{\frac{2023^2}{1012} - 2023, \ldots, \frac{2023^2}{1012} - 2023}_{\text{1011 times}}.$$

3. Let W_1 and W_2 be disjoint circles with radii r_1 and r_2, respectively. (*Disjoint* means that W_1 and W_2 are not allowed to intersect, or to be tangent.) Show that there are two distinct concentric circles that are each tangent to both W_1 and W_2 if and only if $r_1 = r_2$.

SOLUTION. First assume that circle W has center T and radius r, point C is outside of circle W, and W is tangent at point A to circle S_1 with center C and radius s_1 and is tangent at point B to circle S_2 with center C and radius $s_2 < s_1$. Because segments $\overline{AT}$ and $\overline{AC}$ are radii of circles W and S_1, respectively, and each segment is perpendicular to the common tangent line of W and S_1 at A, it follows that T, A, and C are collinear. Similarly, T, B, and C are also collinear, so A, B, T, and C all lie on the same line. Thus, $\overline{AB}$ is a diameter of circle W. Then $r = \frac{1}{2} \cdot AB = \frac{1}{2}(CA - CB) = \frac{s_1 - s_2}{2}$. Therefore, the radius of W is determined by the values of s_1 and s_2. If there were two disjoint circles, W_1 and W_2 each tangent to S_1 and S_2, then their radii would both be $\frac{s_1 - s_2}{2}$, and, therefore, be equal. Note that it is not possible for C to be inside either W_1 or W_2 because if it were inside W_1, for example, then S_2 would be internally tangent to W_1 while W_1 would be internally tangent to S_1. In this case W_2 could not touch both S_1 and S_2 without intersecting with W_1, and W_1 and W_2 would not be disjoint.



Now assume that disjoint circles W_1 with center T_1 and W_2 with center T_2 have the same radius r. Let C be any point on the perpendicular bisector of $\overline{T_1T_2}$. Let $d = CT_1 = CT_2$. Then the circles centered at C with radii $d + r$ and $d - r$ are both tangent to both W_1 and W_2.



4. We have a collection of 2023 squares of various sizes and of total area 4. Show that we can use this collection to fully cover a unit square (overlaps are allowed).

SOLUTION. Let us describe a recursive procedure for covering the unit square. On each step we will have a set of *available* squares and a set of *target* squares that need to be covered with the available squares. We start with 2023 available squares of total area 4 and a single target square of area 1. On each step, we proceed as follows:

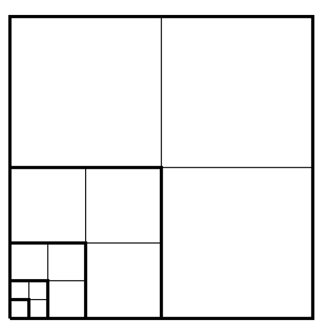
- If one of the available squares is of the same or larger size as one of the target squares, we use it to cover the target square. Each time we cover the largest target square that can be covered with that available square. We now have one fewer available squares and one fewer target squares.

- If all of available squares are smaller than all of the target squares, we take one of the target squares (say, the largest), and divide it into four equal smaller target squares. Thus, we replace one of the target squares by four target squares, each of half its size.

In either case, we obtain new sets of target squares and available squares.

We now continue iterating this operation. The process can terminate in one of two ways: either we run out of target squares (which means we have succeeded) or we run out of available squares, but there are still target squares remaining (we have failed). The process cannot continue indefinitely, because as we subdivide the target squares, they will get smaller and smaller, so eventually they will be small enough to be covered by one of the available squares.

Let us show that it is impossible for the process to fail. We claim that whenever we use an available square to cover a target square, the area of the available square is less than or equal to four times the area of the target square. Otherwise, the available square could have been (and should have been) used on the previous iteration. Since in the beginning, the total area of available squares is four times the total area of target squares, it is impossible to run out of available squares and still have target squares remaining.



ALTERNATE SOLUTION: If one of our squares is at least as large as the unit square then we can use it to cover the unit square. Hence, we can assume that all our squares are smaller than 1×1.

Imagine that the squares in the collection are made out of paper. Take a pair of scissors and cut the squares in the following way. If the side of the square is ℓ with $\frac{1}{2^{n-1}} > \ell \geq \frac{1}{2^n}$ for some $n \geq 1$ then cut it to the size $\frac{1}{2^n} \times \frac{1}{2^n}$. (If $\ell = \frac{1}{2^n}$, then the square doesn't need to be cut). Note that after that procedure, the area of each square decreased at most four times, and, therefore, the total area of the squares in the new collection is at least 1.

Since the covering of the unit square allows overlapping squares, it is enough to show that the unit square can be covered with the new squares because if such a covering is found, then the larger squares, before the cuts, would have also covered the unit square.

To show that the unit square can be covered with the squares from the new collection, do the following. If the collection has 4 squares of the same size $\frac{1}{2^n} \times \frac{1}{2^n}$, then one can make a square of the larger size $\frac{1}{2^{n-1}} \times \frac{1}{2^{n-1}}$ out of these four, by gluing them together in a 2×2 pattern. Starting with the smallest size, we can do this until there are at most 3 squares of the smallest size left. After that we do the same for the second smallest size and so on. This process cannot continue forever since the number of squares is decreasing at each 'gluing' step. When the process is finished, there are the following possibilities.

– We have at some point obtained a square of the size 1×1. We can use this to cover the unit square.

– All of our squares are smaller than 1×1. We will show that this cannot happen.

Suppose that all of our squares are smaller than 1×1 at the end of the 'gluing' steps. Then we have at most 3 squares of each size $\frac{1}{2^n} \times \frac{1}{2^n}$, $n = 1, 2, 3, \ldots, k$ for some finite k. (Where $\frac{1}{2^k}$ is the size of the smallest square at the beginning of the gluing procedure.) Let A be the sum of the area of these squares, by our previous assumptions we have $A \geq 1$, and

$$A \leq 3B, \text{ where } B = \frac{1}{4} + \frac{1}{16} + \ldots \frac{1}{4^k}.$$

But

$$4B = 1 + \frac{1}{4} + \frac{1}{16} + \ldots \frac{1}{4^{k-1}} = 1 + B - \frac{1}{4^k},$$

which implies $3B \leq 1 - \frac{1}{4^k}$, and $B < 1/3$. This leads to $1 \leq A \leq 3B < 1$ which is a contradiction.

Note that both proofs also show that the statement remains true with any number of squares, not just 2023.

5. (This problem concerns the same game as Problem 5 from the previous problem set.)
Two players, Angelica and Brian, each have a rectangular 20×21 board and 140 identical 1×3 tiles ('straight trominos'), enough to completely cover their board with no overlaps. First, Angelica covers her board with her tiles (with no overlaps), and then Brian looks at what she did and covers his board with his tiles. Angelica gets a point for each tile that is in exactly the same position in the two tilings. Show that Brian has a strategy such that Angelica gets at most 14 points for every one of Angelica's tilings.

SOLUTION. Let our board be 20 in height and 21 in width. Let Brian cover the first 2 rows with horizontal tiles. This could be a source of Angelica's 14 points. Divide the remaining 18×21 board into 3×3 squares. In each 3×3 square, if Angelica's corresponding square contains a whole horizontal tile, Brian will cover that 3×3 square with 3 vertical tiles, and if not, Brian will cover it with 3 horizontal tiles. Then the 3×3 squares will not have any matches, and since Brian's trominos within the 18×21 part are always fully inside one of these 3×3 squares. Hence Angelica cannot get any points from this portion of the board, hence she will receive no more than 14 points in total.