WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET III (2022-2023)

1. The Bureau of Weights and Measures had a collection of weights of 1 gram, 2 grams, ..., 2023 grams. When the Bureau moved to a new building, the 100 gram weight was lost. Is it possible to distribute the remaining weights between two storage cabinets so that the total weight in each cabinet is the same?

SOLUTION.

Yes, it is possible. There are many ways to distribute the weights, here is one of them.

Set the 1g and 2g weights aside and split the remaining weights in pairs as follows. First make 48 pairs: (3g, 98g), (4g, 97g), ..., (50g, 51g). Note that each pair weighs the same (101g), and we can put 24 of them in the first cabinet and 24 in the second. After that, make a pair (99g, 2023g) and 961 more pairs (101g, 2022g), (102g, 2021g), ..., (1061g, 1062g). In this set of pairs the first pair weighs 1g less than the next 961 pairs. Put the first pair and 480 more pairs into the first cabinet and the remaining 481 pairs into the second. After that the weights in the second cabinet total 1g more than those in the first cabinet. To make them even, put the 1g weight into the second cabinet and 2g into the first cabinet.

2. Find the greatest positive integer n so that the number $n! \cdot 2022!$ can be expressed as $k!$ for some positive integer k.

SOLUTION. If $n = 2022! - 1$, then $n! \cdot 2022! = (2022!)!$, so $k = 2022!$. This is the greatest possible n that works. Indeed, if $n!(2022!) = k!$, then $k! > n!$, so $k \geq n+1$. Thus, $n!(2022!) = k! \geq (n+1)!$, and dividing both sides of $n!(2022!) \geq (n+1)!$ by $n!$ gives $2022! \geq n+1$. Therefore, $2022! - 1$ is the greatest possible value of n.

3. The difference between the roots of a quadratic polynomial $P(x)$ is equal to 2, and the difference between the roots of $P(x) + 3$ is equal to 4. Find the difference between the roots of $P(x) + 8$.

SOLUTION. First of all, let us complete the square:

$$P(x) = ax^2 + bx + c = a(x + B)^2 + C.$$

(Here $B = \frac{b}{2a}$ and $C = c - \frac{b^2}{4a^2}$, but this is not important. Also, $(-B, C)$ is the apex of the parabola $y = P(x)$.) Changing the coordinate from x to $x + B$ does not change the difference between the roots of a polynomial (both roots shift by the same amount, namely, by B). We can, therefore, assume that $P(x)$ is of the form $P(x) = ax^2 + C$.

Now, the roots of $P(x)$ are symmetric about the origin: $x = \pm\sqrt{-\frac{C}{a}}$. We are told that the difference between the roots equals 2, hence the roots must be ± 1, so that $-\frac{C}{a} = 1$ and $C = -a$. Similarly, the roots of $P(x) + 3 = ax^2 + (C + 3)$ equal $\pm\sqrt{-\frac{C+3}{a}}$, and for the difference to be equal to 4, we must have $-\frac{C+3}{a} = 4$ and $C + 3 = -4a$. Solving the two equations, we obtain $a = -1$, $C = 1$, and $P(x) = 1 - x^2$. Therefore, the roots of $P(x) + 8 = 9 - x^2$ will be ± 3, and the difference equals 6.

Alternate solution: The solutions of $P(x) = ax^2+bx+c$ are given by $\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\frac{-b-\sqrt{b^2-4ac}}{2a}$, so their difference is $\frac{\sqrt{b^2-4ac}}{|a|}$. (You need the absolute value in the denominator because a may be negative.) This leads to the equations

$$\sqrt{b^2-4ac} = 2|a|, \quad \text{and} \quad b^2-4ac = 4a^2.$$

The quadratic polynomial $P(x)+3 = ax^2+bx+c$ has roots $\frac{-b\pm\sqrt{b^2-4a(c+3)}}{2a}$ so the difference of the roots is $\frac{\sqrt{b^2-4a(c+3)}}{|a|}$. leading to the equations

$$\sqrt{b^2-4a(c+3)} = 4|a|, \quad \text{and} \quad b^2-4a(c+3) = 16a^2.$$

This leads to

$$12a^2 = 16a^2 - 4a^2 = (b^2-4a(c+3)) - (b^2-4ac) = -12a,$$

and $12a^2 = -12a$. The solutions of this equation are $a=-1$ and $a=0$, but $a=0$ is not allowed as $P(x)$ then would not be quadratic. Hence, $a=-1$ and

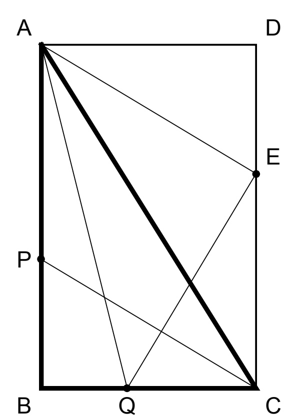
$$b^2-4ac = b^2+4c = 4a^2 = 4.$$

Similarly to the previous computations the difference of the roots of $P(x)+8 = ax^2+bx+c+8$ is given by $\frac{\sqrt{b^2-4a(c+8)}}{|a|}$. Using $a=-1$ and $b^2+4c=4$ we get

$$\frac{\sqrt{b^2-4a(c+8)}}{|a|} = \sqrt{b^2+4c+32} = \sqrt{4+32} = \sqrt{36} = 6.$$

4. $\triangle ABC$ is a right triangle with a right angle at B. The point P is on the side $\overline{AB}$ and the point Q on the side $\overline{BC}$ so that $AP = CB$ and $BP = CQ$. Show that the angle between $\overline{AQ}$ and $\overline{CP}$ is equal to $45°$.

SOLUTION.



Let D be the intersection of the line through A parallel to $\overline{BC}$ and the line through C parallel to $\overline{AB}$ making $\triangle ABC$ into rectangle $ABCD$. Choose the point E on the side $\overline{DC}$ so that $DE = BP$. Then the triangles $\triangle PBC$ and $\triangle ADE$ are congruent since $AD = BC$, $PB = DE$, and $\angle PBC = \angle ADE$. Since the angles of these two triangles agree, we get that $\overline{AE} \parallel \overline{PC}$, which implies that $APCE$ is a parallelogram, and, in particular, $AP = EC$.

Then $\triangle PBC$ (and, hence, $\triangle ADE$) is also congruent to the triangle $\triangle QCE$ since $CQ = BP$, $EC = AP = BC$, and $\angle QCE = \angle PBC$. From these we get that $AE = QC$ and $\angle AED + \angle QCE = \angle QEC + \angle QCE = 90°$. That shows that $\triangle AEC$ is an isosceles right triangle (with $\angle QEA = 90°$), hence, the angle between $\overline{AQ}$ and $\overline{AE}$ is equal to $45°$. But $\overline{AE}$ and $\overline{PC}$ are parallel (since $APCE$ is a parallelogram), which means that the angle between $\overline{AQ}$ and $\overline{PC}$ is also $45°$.

5. Two players, Angelica and Brian, each have a rectangular 20×21 board and 140 identical 1×3 tiles ('straight trominos'), enough to completely cover their board with no overlaps. First, Angelica covers her board with her tiles (with no overlaps), and then Brian looks at what she did and covers his board with his tiles. Angelica gets a point for each tile that is in exactly the same position in the two tilings. Show that Angelica has a strategy for getting at least 14 points regardless of Brian's tiling.

SOLUTION. Let our board be 20 in height and 21 in width. Let Angelica cover her board with horizontal tiles. Let us show that at least 14 tiles on Brian's board must be in the same positions.

Consider Brian's board, and, for each of the 21 column, denote by v_i the number of vertical tiles in this column, and by h_i the number of horizontal tiles whose *leftmost* square is in this column. Here $i = 1, \ldots, 21$ is the number of the column. Thus, each vertical tile counted in v_i covers three squares in the i-th column, while each horizontal tile counted in h_i covers one square in three adjacent columns: i-th, $(i+1)$-st, and $(i+2)$-nd. Notice that $h_{20} = h_{21} = 0$: the corresponding tiles would not fit on the board.

Since Brian's board is completely covered, we get the following conditions:

$$\begin{cases} 3v_1 + h_1 = 20 \\ 3v_2 + h_1 + h_2 = 20 \\ 3v_3 + h_1 + h_2 + h_3 = 20 \\ \ldots \\ 3v_i + h_{i-2} + h_{i-1} + h_i = 20 \\ \ldots \\ 3v_{21} + h_{19} = 20. \end{cases}$$

The tiles matching those on Angelica's board are horizontal tiles whose leftmost square is in the column number 1, or 4,..., or 19. Accordingly, we must prove that

$$h_1 + h_4 + h_7 + h_{10} + h_{13} + h_{16} + h_{19} \geq 14.$$

Let us now find the remainders of h_i's when divided by 3. From the first equation, we see that h_1 has remainder 2. Now the second equation implies that h_2 has remainder 0. Continuing, we see that each of $h_1, h_4, h_7, h_{10}, h_{13}, h_{16}$, and h_{19} have remainder 2, while all the remaining h_i's are divisible by 3 (that is, have remainder zero). Hence, each term in the sum

$$h_1 + h_4 + h_7 + h_{10} + h_{13} + h_{16} + h_{19}$$

is at least 2, implying that the sum is at least 14, as claimed.