WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

PROBLEM SET II (2022-2023) **November 2022**

1. John picked an arbitrary positive integer, multiplied it by 5, multiplied the result by 5, then multiplied the result by 5 again and so on. Is it true that from some point on, all the numbers that John obtains contain a 5 in their decimal representation?

2. Call a collection of different positive integers between 1 and 999 *good* if 1000 can be represented as a sum of several different integers from the collection. Call such a collection *bad* if it is not good. For instance, $\{1, 2, 299, 300, 400\}$ is a good collection, because $1 + 299 + 300 + 400 = 1000$, but $\{5, 6, 7, 8, 9\}$ is a bad collection. What is the maximum possible number of integers in a bad collection?

3. Find the least positive integer n so that the product of all of the positive divisors of n (including n itself) is equal to n^{199}.

4. Let m, n, r, and s be positive integers satisfying $m + n = r + s$ and $r < m \leq n < s$. Show that

$$(2^r + 3^r)(2^s + 3^s) > (2^m + 3^m)(2^n + 3^n).$$

5. The five sides of a convex pentagon $ABCDE$ all have the same length, and the three diagonals $\overline{AD}$, $\overline{AC}$, and $\overline{BE}$ all have the same length. Is it necessary that the pentagon is regular?

You are invited to submit a solution even if you get just one problem. Please do not write your solutions on this problem page. Remember that solutions require a proof or justification.

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WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET II (2022-2023)

1. John picked an arbitrary positive integer, multiplied it by 5, multiplied the result by 5, then multiplied the result by 5 again and so on. Is it true that from some point on, all the numbers that John obtains contain a 5 in their decimal representation?

SOLUTION. The answer is yes. An integer multiple of 5 ends in a 0 or 5, and an odd multiple of 5 will always end in a 5. This shows that if the original number is odd then after the first step it will always end with a 5 (since it will stay odd and will be a multiple of 5).

To consider the even case, first let us write the original number as $2^k m$ where m is odd. (2^k is just the largest power of 2 that divides our number.) After $k + \ell$ multiplications the product becomes $10^k \cdot 5^\ell m$, which is the same as $5^\ell m$ with k additional zeros at the end. For $\ell \geq 1$ the number $5^\ell m$ is an odd multiple of 5, hence it will end in a 5. But this means that after $k + 1$ multiplications we will always have the digit 5 in the decimal representation of our number.

2. Call a collection of different positive integers between 1 and 999 *good* if 1000 can be represented as a sum of several different integers from the collection. Call such a collection *bad* if it is not good. For instance, $\{1, 2, 299, 300, 400\}$ is a good collection, because $1 + 299 + 300 + 400 = 1000$, but $\{5, 6, 7, 8, 9\}$ is a bad collection. What is the maximum possible number of integers in a bad collection?

SOLUTION. Let us prove that the maximum possible number of integers in a bad collection is $N = 500$. To show that $N \geq 500$, notice that the collection $\{500, 501, ..., 999\}$ has 500 integers in it and is bad because the sum of any two integers from this collection is already greater than 1000.

Now let us show that $N \leq 500$. Indeed, consider the 499 sets $\{k, 1000-k\}$, where $1 \leq k \leq 499$, along with the one set $\{500\}$. A bad collection containing more than 500 integers would have to contain both integers from at least one of the 2-element sets, and those two elements add to 1000. Hence every collection of more than 500 integers is good.

3. Find the least positive integer n so that the product of all of the positive divisors of n (including n itself) is equal to n^{199}.

SOLUTION. The wording of this problem was a bit careless, it should have been stated that n is at least 2, otherwise $n = 1$ would have been the smallest such integer. The solution below is for the $n \geq 2$ version. (Students who submitted a solution with $n = 1$ will get full credit.)

Let us agree, once and for all, that "divisors of n" means "positive divisors of n, including 1 and n itself".

Note that if x is a divisor of n, then so is n/x, and that the product $n \cdot (n/x) = n$. In other words, divisors of n occur in pairs with each pair having product n. The only exception is if n is a perfect square, in which case the divisor $\sqrt{n}$ is left without a pair.

Thus, there are two cases:

- n is not a perfect square: the number of divisors is even, say, $2m$, they form k pairs, and the product of all the divisors is n^m.

- n is a perfect square: the number of divisors is odd, say, $2m+1$, they form m pairs and the divisor $\sqrt{n}$ is without a pair, and the product of all the divisors is $n^m \cdot \sqrt{n}$.

In either case, the product of all the divisors is $n^{D/2}$, where D is the number of divisors of n. Thus, we seek the least n that has exactly $2 \cdot 199 = 398$ divisors.

The number of divisors of n is determined by its prime factorization. If n is written as the product

$$n = p_1^{d_1} \cdot p_2^{d_2} \ldots p_k^{d_k}$$

for distinct primes $p_1, \ldots, p_k$, we obtain a divisor of n by taking a product of p_1 with exponent at most d_1, p_2 with exponent at most d_2, and so on. That is, there are $d_i + 1$ choices (namely, $0, 1, \ldots, d_i$) for the exponent of each of the primes p_i. Because these exponents can be chosen independently, the total number of divisors of n equals

$$(d_1+1)(d_2+1)\ldots(d_k+1).$$

In this problem, m must have 398 divisors. Since 199 is prime, there are only two ways to obtain this number as a product (up to reorder), giving two cases:

- n has a single prime factor with exponent 397: $n = p_1^{397}$, and its number of divisors is $(397+1) = 398$. Clearly, the least such n is 2^{397}.

- n has two prime factors with exponents 198 and 1: $n = p_1^{198} p_2$, and its number of divisors is $(198+1)(1+1) = 398$. Clearly, the least such n is $2^{198} \cdot 3$.

The second case gives a much smaller number, so the answer is

$$2^{198} \cdot 3 = 1205203533194242706656471569255871951891652245337094626476032.$$

(Which, in case you are wondering, reads 1 novemdecillion 205 octodecillion 203 septendecillion 533 sexdecillion 194 quindecillion 242 quattuordecillion 706 tredecillion 656 duodecillion 471 undecillion 569 decillion 255 nonillion 871 octillion 951 septillion 891 sextillion 652 quintillion 245 quadrillion 337 trillion 94 billion 626 million 476 thousand 32.)

4. Let m, n, r, and s be positive integers satisfying $m+n = r+s$ and $r < m \leq n < s$. Show that

$$\left(2^r+3^r\right)\left(2^s+3^s\right) > \left(2^m+3^m\right)\left(2^n+3^n\right).$$

SOLUTION. Note that

$$\begin{aligned}
&\left(2^r+3^r\right)\left(2^s+3^s\right) - \left(2^m+3^m\right)\left(2^n+3^n\right) \\
=\ &\left(2^{r+s}+3^{r+s}\right) + \left(2^r 3^s + 2^s 3^r - 2^m 3^n - 2^n 3^m\right) - \left(2^{m+n}+3^{m+n}\right) \\
=\ &2^r 3^s + 2^s 3^r - 2^m 3^n - 2^n 3^m \\
=\ &2^s 3^r \left(1 + \left(\frac{3}{2}\right)^{s-r} - \left(\frac{3}{2}\right)^{n-r} - \left(\frac{3}{2}\right)^{m-r}\right) \\
=\ &2^s 3^r \left(\left(\frac{3}{2}\right)^{n-r} - 1\right)\left(\left(\frac{3}{2}\right)^{m-r} - 1\right).
\end{aligned}$$

Because both $n-r$ and $m-r$ are positive, and $\frac{3}{2} > 1$, this product is positive implying the needed inequality.

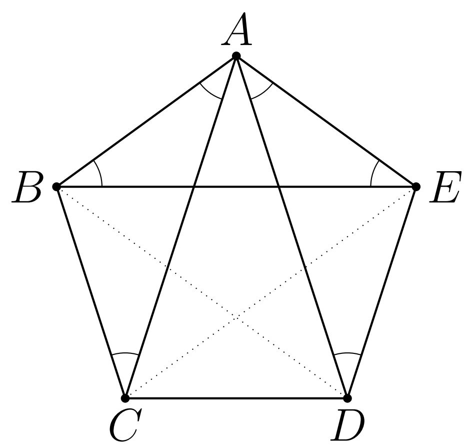
5. The five sides of a convex pentagon $ABCDE$ all have the same length, and the three diagonals $\overline{AD}$, $\overline{AC}$, and $\overline{BE}$ all have the same length. Is it necessary that the pentagon is regular?

SOLUTION. The triangles $\triangle ABC$, $\triangle EAB$, and $\triangle DAE$ are congruent (two of their sides are equal sides of the pentagon, and one side is one of the equal diagonals $AC = BE = DA$). Then the corresponding angles are the same as well; in particular,

$$\angle ABC = \angle DEA = \angle EAB,$$

and

$$\angle ABE = \angle AEB = \angle BAC = \angle BCA = \angle EAD = \angle EDA.$$



Triangles $\triangle ABD$ and $\triangle EDB$ are congruent because $AB = DE$, $AD = BE$, and $\overline{BD}$ is a shared side. Thus, $\angle ABD = \angle EDB$. Also, because $\triangle CBD$ is isosceles with $BC = CD$, $\angle CBD = \angle CDB$. Thus, $\angle ABC = \angle ABD + \angle CBD = \angle BDE + \angle CDB = \angle CDE$. Similarly, $\angle AED = \angle BCE$. Together with $\angle ABC = \angle DEA = \angle EAB$ this shows that all five of the internal angles in the pentagon are equal, hence the pentagon is regular.

ANOTHER SOLUTION. We will show that the vertices of the pentagon are on the same circle. If the center of that circle is O, then each side has the same central angle $\frac{360^o}{5} = 72^o$. Connecting O to two neighboring vertex of the pentagon creates an isosceles triangle with two sides equal to the radius of the circle, and the angle at O is 72^o. Hence, the other two angles are $\frac{180^o - 72^o}{2} = 54^o$. But this shows that each angle of the pentagon is equal to $2 \cdot 54^o = 108^o$, which proves that the pentagon is regular.

To show that the vertices of the pentagon are on the same circle we first prove that A, B, C, and E are on the same circle. Note that $\triangle ABC$ and $\triangle ABE$ are congruent ($\overline{AB}$ is a shared side, $BC = AE$ and $AC = BE$), hence, $\angle BCA = \angle BEA$. By the Inscribed Angle Theorem it follows that A, B, C, and E are on the same circle. The same argument shows that B, A, E, and D are also on the same circle. But these two circles both go through A, B, and E. Hence, the two circles are the same, and they contain all five vertices of the pentagon.