# WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

## SOLUTIONS TO PROBLEM SET I (2022-2023)

**1.** The last Badger game had 10% more people in attendance than the previous game, but the number of empty seats was 15% less than during the previous game. What percentage of seats was occupied during the last Badger game, assuming all the people in attendance took their seats during the game.

**SOLUTION.** Let $N$ be the stadium capacity, and $M$ be the number of occupied seats during the previous game (that is, $N - M$ empty seats). Based on our information, during the last game we had $1.1M$ occupied seats and $0.85(N - M)$ empty seats. So

$$1.1M + 0.85(N - M) = N \quad \Leftrightarrow \quad 0.25M = 0.15N \quad \Leftrightarrow \quad M = 0.6N,$$

and on the last game the proportion of seats occupied was

$$\frac{1.1M}{N} = 0.66,$$

which shows that 66% of the seats were occupied.

**2.** Let $n > 2$ be an integer, and consider the fractions $\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}$. Show that the number of irreducible fractions in this collection is even. (A fraction is called irreducible if the numerator and the denominator do not have common integer divisors other than $\pm 1$.)

**SOLUTION.** Notice that if a fraction $k/n$ is irreducible, then so is the fraction $(n-k)/n$. Indeed, if $(n - k)/n$ is not irreducible, then both $n - k$ and $n$ are divisible by some integer $l > 1$, that is, there are integers $m$ and $p$ so that $n - k = lm$ and $n = lp$. Then $k = n - (n - k) = l(p - m)$ is also a multiple of $l$, and $k/n$ would not be irreducible. Also, if $k = n - k$, then $n = 2k$ and $k/n = k/2k$ is reducible, because $k = n/2 > 1$. Therefore, all irreducible fractions split into pairs, $k/n, (n - k)/n$. It follows that the number irreducible fractions is 2 times the number of pairs.

**3.** Show that if $x, y$, and $z$ are nonzero numbers satisfying the equations

$$\frac{x+y}{x^2+y^2} = \frac{y+z}{y^2+z^2} = \frac{z+x}{z^2+x^2},$$

then we must have $x = y = z$.

**SOLUTION.** The easiest way to show this is to use the fact that if $b$, $d$, and $b + d$ are not zero, then $\frac{a}{b} = \frac{c}{d}$ implies that $\frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d}$. This follows from the fact that $ad = bc$ implies that $ad + ab = bc + ab$, so $a(b + d) = b(a + c)$ and $\frac{a}{b} = \frac{a+c}{b+d}$. Given this,

$$\frac{x+y}{x^2+y^2} = \frac{y+z}{y^2+z^2} = \frac{x+2y+z}{x^2+2y^2+z^2}.$$

Then

$$\frac{x+y}{x^2+y^2} = \frac{-(z+x)}{-(z^2+x^2)} = \frac{x+2y+z-(z+x)}{x^2+2y^2+z^2-(z^2+x^2)} = \frac{2y}{2y^2} = \frac{1}{y}.$$

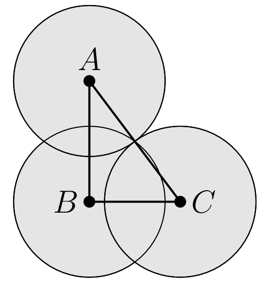
By symmetry, the given fractions are equal to $\frac{1}{x} = \frac{1}{y} = \frac{1}{z}$, and the needed result follows.

Alternatively, cross multiplying

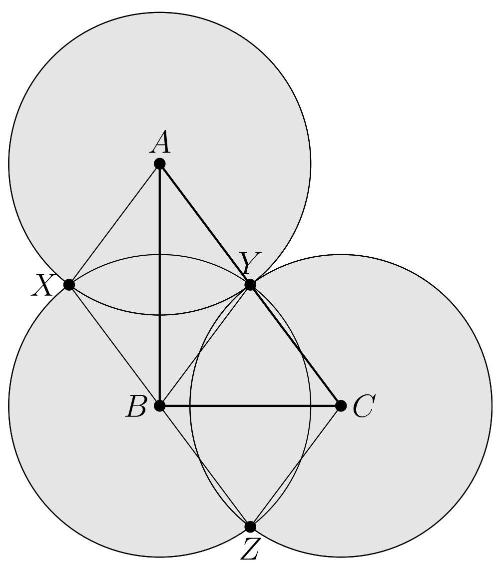
$$\frac{x+y}{x^2+y^2} = \frac{y+z}{y^2+z^2} \quad \text{and} \quad \frac{y+z}{y^2+z^2} = \frac{z+x}{z^2+x^2}$$

and adding the two resulting equations simplifies to $yz^2 = zy^2$. Therefore, $0 = yz^2 - zy^2 = yz(z-y)$, and because the variables are nonzero, this implies $y = z$. By symmetry, $x = y$ and $x = y = z$ as needed.

**4.** Triangle $\triangle ABC$ has sides of lengths $AB = 16$, $BC = 12$, and $CA = 20$. Three circles, all of radius 10, are centered at points $A$, $B$, and $C$. Find the total area covered by the three circles (the shaded area in the picture).




**SOLUTION.** First of all, notice that $\triangle ABC$ is a right triangle, because $AB^2 + BC^2 = AC^2$. Let $S_A$, $S_B$, and $S_C$ be the circles centered at $A$, $B$, and $C$, respectively. Denote by $Y$ the midpoint of the hypotenuse $\overline{AC}$. Since $AY = YC = 10$, which is exactly the radius of the circles, the circles $S_A$ and $S_C$ pass through $Y$. Moreover, they are tangent to each other at this point, because the distance between their centers equals twice their radius. Since $\triangle ABC$ is a right triangle, $BY = AY$, so the circle $S_B$ also passes through $Y$. (All of this is clear on the picture, but we still want to confirm it.)




The circles $S_A$ and $S_B$ meet at the point $Y$; let $X$ be their second intersection point. Similarly, $S_C$ and $S_B$ also meet at $Y$; denote their other intersection point by $Z$. The triangles $\triangle AXB$ and $\triangle AYB$ are congruent isosceles triangles, so $\angle BAY = \angle ABY = \angle ABX$, and hence, $\overline{AY} \parallel \overline{XB}$. Arguing in the same way, we see that the $\triangle BZC$ and $\triangle BYC$ are congruent isosceles triangle, and that $\overline{YC} \parallel \overline{BZ}$.

We now see that the three points $X$, $B$, and $Z$ are collinear because $\overline{XB}$ and $\overline{BZ}$ are both parallel to the line $AC$. In other words, $\overline{XZ}$ is a diameter of the circle $S_B$. Besides, the quadrilateral $XACZ$ is a parallelogram. Its area is twice the area of the original $\triangle ABC$ because of the congruence of the two pairs of triangles $\triangle AXB$ and $\triangle AYB$, and $\triangle BZC$ and $\triangle BYC$. Hence, its area is $2 \cdot \frac{12 \times 16}{2} = 192$.

Finally, the area we need to find is composed of the parallelogram $XACZ$ and three sectors. The sector of the circle $S_A$ (that is, the part of the circle that is outside of the parallelogram) has central angle $360° - \angle XAY = 360° - 2\angle BAC$. Similarly, the sector of the circle $S_C$ has central angle $360° - \angle ZCY = 360° - 2\angle BCA$. Since $\triangle ABC$ is a right triangle, $\angle BAC + \angle BCA = 90°$, and the total angular measure of the two sectors is $2 \times 360° - 180° = 540°$. Finally, the sector of the circle $S_B$ is a semicircle (of angular measure $180°$, meaning that the three sectors have total angle $720°$, twice the complete angle). We now see that the total area of the shaded region equals the area of the parallelogram (192) plus the area of three sectors that total two full circles of radius 10. Thus, the answer is

$$192 + 2\pi \cdot 10^2 = 192 + 200\pi.$$

**5.** Someone chose 8 different positive integers no greater than 16. Show that among all of their positive pairwise differences there are at least 3 that are equal.

**SOLUTION.** We will show the statement by contradiction. Let us assume that there is a way to choose 8 numbers so that we cannot find three pairwise differences that are equal.

Denote the 8 numbers in order by $1 \leq a_1 < a_2 < \cdots < a_8 \leq 16$. Consider the positive differences $a_2 - a_1, a_3 - a_2, \ldots, a_8 - a_7$. Note that the sum of these seven numbers is

$$(a_8 - a_7) + (a_7 - a_6) + \cdots + (a_2 - a_1) = a_8 - a_1 \leq 16 - 1 = 15.$$

Denote these 7 differences (all positive integers) in order by $1 \leq b_1 \leq b_2 \leq \cdots \leq b_7$. By assumption, each number can only show up at most twice among these seven, which means that $b_3$ must be at least 2, otherwise $b_1 = b_2 = b_3$. Hence, $2 \leq b_3 \leq b_4 \leq \cdots \leq b_7$. Similarly, $b_5$ is at least 3 (otherwise $b_3 = b_4 = b_5 = 2$), and $b_7$ is at least 4 (with a similar argument). Hence,

$$b_1 + b_2 + b_3 + \cdots + b_7 \geq 1 + 1 + 2 + 2 + 3 + 3 + 4 = 16.$$

But this is a contradiction, as we have seen that the sum of these seven numbers is at most 15. Hence, our assumption was incorrect: there must be at least three pairwise differences that are equal.