WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET V (2021-2022)

1. In how many ways can you mark 8 of the 16 cells in a 4×4 grid of cells so that there are 2 marked cells in each row, 2 marked cells in each column, and neither of the diagonals have all 4 of their cells marked?

SOLUTION. The answer is 73. Number the rows of the grid from top to bottom with 1, 2, 3, and 4, and number the columns from left to right with 1, 2, 3, and 4. First consider the number of ways there are to mark 8 cells with 2 marked cells in each row and each column. Then the 2 marked rows for any particular column of the grid must be one of the pairs 12, 13, 14, 23, 24, or 34. Consider the cases where no 2 columns have marks in the same set of 2 rows and the cases when there are 2 such columns.

- If no 2 columns have marks in the same set of 2 rows, then the columns use 4 of 6 row pairs 12, 13, 14, 23, 24, or 34. Because row 1 must be marked in exactly 2 columns, the columns must use exactly 2 of the row pairs 12, 13, or 14 (and 2 of the other 3 row pairs). Once cells have been marked in 2 columns using 2 of the pairs of 12, 13, 14, there will be one row out of 2, 3, and 4 that has no marks. For each of the rows 2, 3, and 4 there is exactly one way of choosing two of the pairs 23, 24, and 34 so that there are two marked cells in that row from these pairs (for example, for row 2 the other pairs must be 23 and 24). Hence, the 2 row pairs from 12, 13, and 14 determine the two other chosen pairs from 23, 24, and 34.

  There are 3 ways of selecting 2 of the pairs 12, 13, and 14, hence there are 3 ways of choosing the 4 different possible pairs of 12, 13, 14, 23, 24,and 34 that can be used in this case. For each choice of 4 pairs there are $4 \cdot 3 \cdot 2 \cdot 1 = 4!$ ways to assign the pairs to the 4 columns. Thus, there are $3 \cdot 4! = 72$ ways to mark the cells in this case.

- If 2 columns are marked in the same set of 2 rows, then it must be that the other 2 columns are both marked in the other 2 rows. There are 6 ways to choose the marked rows for column 1, and 3 ways to choose another column with marks in those same rows. Once these choices are made, the marked rows for the other two columns are determined. Thus, there are $6 \cdot 3 = 18$ ways to place the marks in this case.

Therefore, there are $72 + 18 = 90$ ways to mark 8 cells so there are 2 marked cells in each row and each column.

Now consider the number of ways to mark the cells where all the cells in at least one diagonal are marked. If all the cells of the main diagonal of the grid are marked, then column 1 is marked in rows 1 and a, where a is one of 2, 3, or 4. Column a of the grid will have a mark in row a and one other row.

- If the other row is row b, where $b \neq 1$, then let c be the column number which is not 1, a, or b. Then column b will have to have marks in rows b and c, and column c will have to have marks in rows 1 and c. Thus, there are 3 choices for the value of a and then 2 choices for the value of b, and the rest of the marks are fixed. Hence, there are $3 \cdot 2 = 6$ ways to mark the cells in this way.

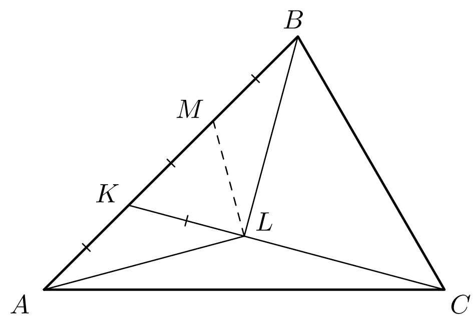
- If the other row is 1, then both columns 1 and a have marks in both rows 1 and a, so the other 2 columns will have marks in the other 2 rows. There are 3 choices for the value of a, so there are 3 ways to place marks in this way.

Thus, there are $6+3=9$ ways to mark the cells so that marks appear on the main diagonal of the grid. Similarly, there are 9 ways to mark the cells so that marks appear on the other diagonal of the grid. Because there is 1 way to place marks so that all the cells of both diagonals are marked, there is a total of $9+9-1=17$ ways to mark the cells so that all the cells on at least one of the diagonals are marked. Therefore, the requested number of ways to mark 8 cells is $90-17=73$.

2. A triangle ABC has a point K on the side AB and a point L on the segment CK such that $AK = KL = \frac{1}{2}KB$. Show that if $\angle CAB = 45^\circ$ and $\angle CKB = 60^\circ$, then $AL = BL = CL$.

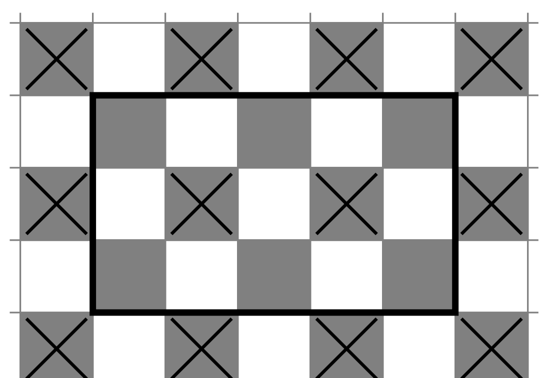
SOLUTION. Triangle AKL is an isosceles triangle with external angle $\angle BKL = \angle BKC = 60^\circ$, so the two base angles are $\angle KAL = \angle KLA = 30^\circ$. Triangle ALC has external angle $\angle ALK = 30^\circ$, and angle $\angle LAC = \angle CAB - \angle LAB = 45^\circ - 30^\circ = 15^\circ$, so $\angle LCA = 15^\circ$ too. As a result, the triangle ALC is isosceles and $AL = CL$.

We now show that $AL = BL$. Let M be the midpoint of the segment BK. Then triangle KLM is an isosceles triangle with a 60° angle, so it is equilateral. Finally, $\triangle AKL$ and $\triangle BML$ are congruent by the side-angle-side congruence criterion: $AK = KL = LM = MB$ and $\angle AKL = 180^\circ - \angle MKL = 180^\circ - \angle KML = \angle BML$, so $AL = BL$.



3. We tiled the plane perfectly with 1×1 unit squares, forming an infinite grid. Taylor removed finitely many of these tiles, with no two of the removed tiles having common vertices or edges. Is it true that the remaining tiles can always be covered with non-overlapping 2×1 dominoes? (Each domino has to cover exactly two unit squares.)

SOLUTION. We will show that Taylor can remove tiles in a way that the remaining part cannot be covered with 2×1 dominoes. Here is one possibility: consider a 5×7 sized part of the original grid, and assume that Taylor removes the eight tiles denoted by $\times$ on the picture below. The removed tiles are the 1st, 3rd, 5th, and 7th in the 1st, 3rd, and 5th row. Note that these squares do not have overlapping vertices or edges, so Taylor can remove them. Color alternate cells with black and white in a checkerboard pattern, with the upper left corner being black. Call the 3×5 rectangle in the middle of the 5×7 rectangle the *small rectangle*.



Suppose we can cover the remaining tiles with dominoes. In particular, each of the white squares in the small rectangle would be covered. A domino that covers a white square in the small rectangle also covers a black square. The black square need not be in the small rectangle, but it definitely would be in the original 5×7 rectangle. However, the small rectangle contains 7 white squares, while the large rectangle has only 6 black squares left (they all happen to be in the small square, Taylor removed the rest). Thus, covering the remaining tiles with dominoes is impossible.

4. The function f is defined for all real x other than $x = 0$ and $x = 1$ and satisfies the equation

$$f(x) + f(1 - \tfrac{1}{x}) = 2^x$$

for all x where it is defined. Find a formula for $f(x)$.

SOLUTION. Let t be any number where the function is defined. We then have

$$f(t) + f\left(1 - \frac{1}{t}\right) = 2^t.$$

We now use the same equation for $x = 1 - \dfrac{1}{t}$. Then

$$1 - \frac{1}{x} = 1 - \frac{1}{1 - \frac{1}{t}} = 1 - \frac{t}{t-1} = -\frac{1}{t-1},$$

and the equation becomes

$$f\left(1 - \frac{1}{t}\right) + f\left(-\frac{1}{t-1}\right) = 2^{1-\frac{1}{t}}.$$

Finally, we use the equation again for $x = -\dfrac{1}{t-1}$. Then

$$1 - \frac{1}{x} = 1 - \frac{1}{\left(-\frac{1}{t-1}\right)} = 1 + (t-1) = t,$$

and the equation becomes

$$f\left(-\frac{1}{1-t}\right) + f(t) = 2^{-\frac{1}{t-1}}.$$

(Note that since $t \neq 0$ or 1, both expressions $x = 1 - \dfrac{1}{t}$ and $x = -\dfrac{1}{t-1}$ are well defined; also, they cannot equal 0 or 1, so we are allowed to use the equation.)

These three equations give a system of three linear equations in the three unknowns $f(t)$, $f\left(1 - \dfrac{1}{t}\right)$, and $f\left(-\dfrac{1}{t-1}\right)$:

$$\begin{cases} f(t) + f\left(1 - \frac{1}{t}\right) & = 2^t \\ f\left(1 - \frac{1}{t}\right) + f\left(-\frac{1}{t-1}\right) & = 2^{1-\frac{1}{t}} \\ f\left(-\frac{1}{t-1}\right) + f(t) & = 2^{-\frac{1}{t-1}}. \end{cases}$$

It remains to solve this system. In fact, we only need to determine $f(t)$.

Adding the three equations gives

$$2f(t) + 2f\left(1 - \frac{1}{t}\right) + 2f\left(-\frac{1}{t-1}\right) = 2^t + 2^{1-\frac{1}{t}} + 2^{-\frac{1}{t-1}},$$

which, divided by 2, gives

$$f(t) + f\left(1 - \frac{1}{t}\right) + f\left(-\frac{1}{t-1}\right) = \frac{1}{2}\left(2^t + 2^{1-\frac{1}{t}} + 2^{-\frac{1}{t-1}}\right).$$

Finally, subtracting the second equation of the original system, gives

$$f(t) = \frac{1}{2}\left(2^t + 2^{1-\frac{1}{t}} + 2^{-\frac{1}{t-1}}\right) - 2^{1-\frac{1}{t}} = \frac{1}{2}\left(2^t - 2^{1-\frac{1}{t}} + 2^{-\frac{1}{t-1}}\right) = 2^{t-1} - 2^{-\frac{1}{t}} + 2^{-\frac{t}{t-1}}.$$

5. Show that if x and y are real numbers, then we can always find integers m and n with

$$(x-m)^2 + (y-n)^2 + (x-m)(y-n) \le \frac{1}{3}.$$

SOLUTION 1.

It is enough to show that for any point (x, y) inside the square $S = \{0 \le x \le 1, 0 \le y \le 1\}$ in the coordinate plane the inequality holds for (m, n) equal to the coordinates of one of the vertices

$$V_1 = (0,0),\ V_2 = (0,1),\ V_3 = (1,1),\ V_4 = (1,0)$$

of the square. Then for an arbitrary point (x, y) one can choose the coordinates of one of the vertices of the unit square with integer coordinates where (x, y) is located.

Consider the following points inside S:

$$A = \left(\tfrac{1}{3}, \tfrac{1}{3}\right),\ B = \left(\tfrac{2}{3}, \tfrac{2}{3}\right)$$

and the midpoints of each side of S:

$$C_1 = \left(0, \tfrac{1}{2}\right),\ C_2 = \left(\tfrac{1}{2}, 0\right), C_3 = \left(\tfrac{1}{2}, 1\right),\ C_4 = \left(1, \tfrac{1}{2}\right).$$

The segments

$$[A, B],\ [A, C_1],\ [A, C_2],\ [B, C_3],\ [B, C_4]$$

split the square S into four parts. Let us denote those parts S_1, S_2, S_3, S_4 so that $V_k \in S_k$, as shown in the picture below.

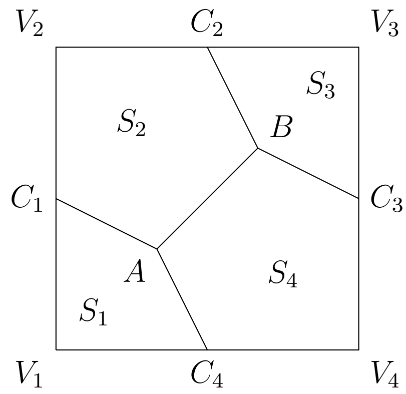
We will show that for all points (x, y) in S_k the inequality holds with $(m, n) = V_k$.

First notice that if the inequality holds for some point $P = (x, y) \in S$ and $(m, n) = V_k$, then it is also holds, with the same (m, n), for all points on the interval from V_k to P. Indeed, if $Q = (x_1, y_1)$ is a point on that interval, then

$$(x_1 - m) = c(x - m) \quad \text{and} \quad (y_1 - n) = c(y - n)$$

for some $0 \le c \le 1$. Therefore,

$$(x_1 - m)^2 + (y_1 - n)^2 + (x_1 - m)(y_1 - m) = c^2((x-m)^2 + (y-n)^2 + (x-m)(y-m)) \le \frac{1}{3}.$$



Solution 1: Parts

Next, notice that to verify the inequality for all points in S_k (and V_k) one only needs to verify it on the boundary of S_k. For all other points the inequality follows from our last argument, because every point in S_k lies on a segment going from V_k to one of the points on the segments on the boundary of S_k ending at A or B.

To show that all points (x, y) in S_1 satisfy the inequality with $(m, n) = V_1 = (0, 0)$ we need to verify it for the points on $[A, C_1]$ and $[A, C_2]$. All points on $[A, C_1]$ satisfy the equation $y = -\frac{x}{2} + \frac{1}{2}$, $0 \le x \le \frac{1}{3}$. The inequality becomes

$$x^2 + \left(-\frac{x}{2} + \frac{1}{2}\right)^2 + x\left(-\frac{x}{2} + \frac{1}{2}\right) \le \frac{1}{3}.$$

After expanding the left-hand side and simplifying it becomes

$$\frac{3}{4}x^2 - x - \frac{1}{12} \le 0.$$

Multiplying both sides by $\frac{1}{2}$ and completing the square it becomes

$$(3x - 2)^2 - 5 \le 0.$$

When x is between 0 and $1/3$, $(3x - 2)^2$ is between 0 and 4. Hence, the inequality holds.

For points on $[A, C_2]$ perform similar estimates or just notice that $(x, y) \in [A, C_2]$ if and only if $(y, x) \in [A, C_1]$ and reduce it to the previous step.

To show that all points (x, y) in S_2 satisfy the inequality with $(m, n) = V_2 = (0, 1)$ we need to verify it for the points on $[A, C_1]$, $[B, C_3]$ and $[A, B]$. Once again, using $y = -\frac{x}{2} + \frac{1}{2}$ for $(x, y) \in [A, C_1]$ the inequality becomes

$$x^2 + \left(-\frac{x}{2} - \frac{1}{2}\right)^2 + x\left(-\frac{x}{2} - \frac{1}{2}\right) \le \frac{1}{3}.$$

After expansion and cancellation it becomes

$$9x^2 \le 1,$$

which holds for $0 \le x \le 1/3$. For $[B, C_3]$ the estimates can be performed similarly, or one can notice that after the change of variables $s = 1 - y$, $t = 1 - x$ one may repeat the previous step. The points on $[A, B]$ satisfy $x = y$, $\frac{1}{3} \le x \le \frac{2}{3}$ and the inequality becomes

$$x^2 + (x - 1)^2 + x(x - 1) \le \frac{1}{3},$$

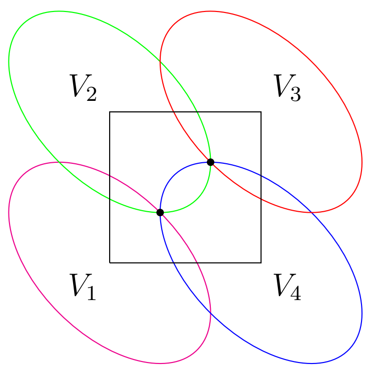
which after expansion in the left-hand side, becomes

$$3x^2 - 3x + \frac{2}{3} \leq 0.$$

The quadratic polynomial on the left has roots at $x = \frac{1}{3}$ and $\frac{2}{3}$, and the inequality holds for $\frac{1}{3} \leq x \leq \frac{2}{3}$.

Finally, the arguments for S_3 and S_4 can be performed similarly, or one may just notice that after the substitution $s = 1 - x$, $t = 1 - y$, the case of S_3 is reduced to the case of S_1 and the case of S_4 to S_2.

SOLUTION 2. Let S and V_k be as above. Those familiar with the basics of analytic geometry may notice that points $(x, y) \in S$ satisfying the inequality with $(m, n) = V_1$ lie inside the ellipse $E_1 = \{x^2 + y^2 + xy \leq \frac{1}{3}\}$ whose center is at $V_1 = (0, 0)$. Points satisfying the inequality with other V_k lie in the ellipses E_2, E_3, and E_4 obtained from E_1 by parallel translation, so that their centers move to V_2, V_3 and V_4, respectively, as shown in the diagram.



Solution 2: Ellipses.

It is easy to verify that E_1, E_2, and E_4 intersect at $\left(\frac{1}{3}, \frac{1}{3}\right)$ and E_2, E_3, and E_4 intersect at $\left(\frac{2}{3}, \frac{2}{3}\right)$. After that one can find the points of intersection of the ellipses with the sides of the square and, using convexity of each ellipse, show that their union covers the square.

SOLUTION 3.

It is enough to show that for any point (x, y) inside the square $S = \{0 \leq x \leq 1, 0 \leq y \leq 1\}$ in the coordinate plane the inequality holds for (m, n) equal to the coordinates of one of the vertices

$$V_1 = (0, 0),\ V_2 = (0, 1),\ V_3 = (1, 1),\ V_4 = (1, 0)$$

of the square. Then for an arbitrary point (x, y) one can choose the coordinates of one of the vertices of the unit square where (x, y) is located.

So, more precisely, we need to show that solutions of 4 inequalities:

$$x^2 + y^2 + xy \leq \frac{1}{3},$$

$$(x-1)^2 + y^2 + (x-1)y \leq \frac{1}{3},$$

$$x^2 + (y-1)^2 + x(y-1) \leq \frac{1}{3},$$

$$(x-1)^2 + (y-1)^2 + (x-1)(y-1) \leq \frac{1}{3}$$

covers all square S with vertices V_1, V_2, V_3 and V_4.

We introduce a new coordinate system (X, Y) which is related with old one (x, y) by mapping

$$\begin{cases} X = \sqrt{3}(x+y) \\ Y = (x-y) \end{cases} \Leftrightarrow \begin{cases} x = \dfrac{\sqrt{3}}{6}X + \dfrac{1}{2}Y \\ y = \dfrac{\sqrt{3}}{6}X - \dfrac{1}{2}Y. \end{cases}$$

In the new coordinate system, V_1, V_2, V_3, and V_4 have coordinates

$$V_1 = (0,0), \quad V_2 = (\sqrt{3}, -1), \quad V_3 = (2\sqrt{3}, 0), \quad V_4 = (\sqrt{3}, 1),$$

and S is the rhombus with vertices V_1, V_2, V_3,and V_4. In the new variables X and Y, the inequalities become

$$\begin{gathered} (\frac{\sqrt{3}}{6}X + \frac{1}{2}Y)^2 + (\frac{\sqrt{3}}{6}X - \frac{1}{2}Y)^2 + (\frac{\sqrt{3}}{6}X + \frac{1}{2}Y)(\frac{\sqrt{3}}{6}X - \frac{1}{2}Y) \le \frac{1}{3}, \\ (\frac{\sqrt{3}}{6}X + \frac{1}{2}Y - 1)^2 + (\frac{\sqrt{3}}{6}X - \frac{1}{2}Y)^2 + (\frac{\sqrt{3}}{6}X + \frac{1}{2}Y - 1)(\frac{\sqrt{3}}{6}X - \frac{1}{2}Y) \le \frac{1}{3}, \\ (\frac{\sqrt{3}}{6}X + \frac{1}{2}Y)^2 + (\frac{\sqrt{3}}{6}X - \frac{1}{2}Y - 1)^2 + (\frac{\sqrt{3}}{6}X + \frac{1}{2}Y)(\frac{\sqrt{3}}{6}X - \frac{1}{2}Y - 1) \le \frac{1}{3}, \\ (\frac{\sqrt{3}}{6}X + \frac{1}{2}Y - 1)^2 + (\frac{\sqrt{3}}{6}X - \frac{1}{2}Y - 1)^2 + (\frac{\sqrt{3}}{6}X + \frac{1}{2}Y - 1)(\frac{\sqrt{3}}{6}X - \frac{1}{2}Y - 1) \le \frac{1}{3}. \end{gathered}$$

After expansion, this becomes

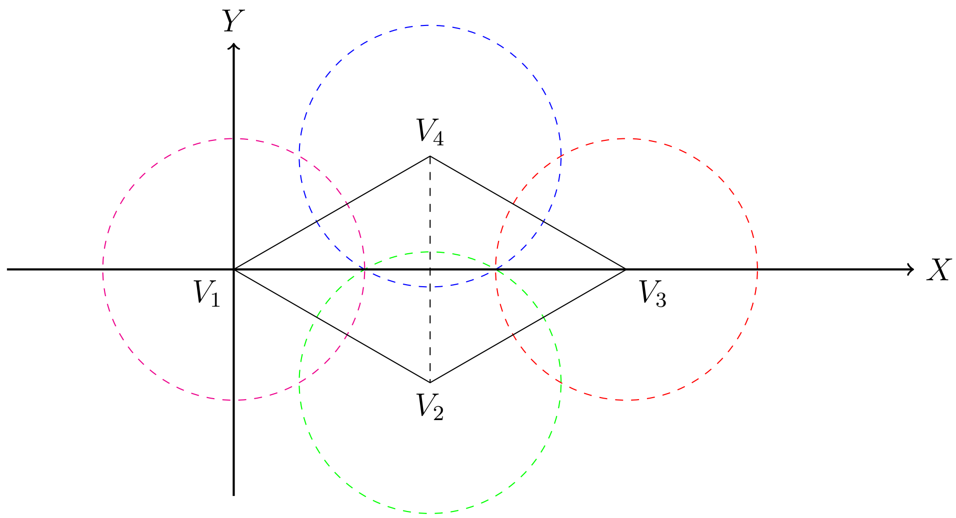
$$\begin{gathered} \frac{1}{4}X^2 + \frac{1}{4}Y^2 \le \frac{1}{3}, \\ \frac{1}{4}X^2 + \frac{1}{4}Y^2 - \frac{\sqrt{3}}{2}X - \frac{1}{2}Y + 1 \le \frac{1}{3}, \\ \frac{1}{4}X^2 + \frac{1}{4}Y^2 - \frac{\sqrt{3}}{2}X + \frac{1}{2}Y + 1 \le \frac{1}{3}, \\ \frac{1}{4}X^2 + \frac{1}{4}Y^2 - \sqrt{3}X + 3 \le \frac{1}{3}. \end{gathered}$$

Multiplying by 4 and completing squares gives us

$$\begin{gathered} X^2 + Y^2 \le \left(\frac{2}{\sqrt{3}}\right)^2, \\ (X - \sqrt{3})^2 + (Y-1)^2 \le \left(\frac{2}{\sqrt{3}}\right)^2, \\ (X - \sqrt{3})^2 + (Y+1)^2 \le \left(\frac{2}{\sqrt{3}}\right)^2, \\ (X - 2\sqrt{3})^2 + Y^2 \le \left(\frac{2}{\sqrt{3}}\right)^2, \end{gathered}$$

which correspond to disks with radius $\frac{2}{\sqrt{3}}$.

Thus, in the new coordinate system, the inequality sets are disks with centers in V_1, V_2, V_3, and V_4, and radius $\frac{2}{\sqrt{3}}$.



Solution 3: New coordinate system

Why do these 4 disks cover rhombus $V_1V_2V_3V_4$? We first show that they cover the $\triangle V_1V_2V_4$. Since we have symmetry on V_2V_4, a similar argument will show that $\triangle V_2V_3V_4$ is also covered, and then the whole rhombus is covered.

The triangle $\triangle V_1V_2V_4$ is equilateral with side length 2, so its circumradius is exactly $\frac{2}{\sqrt{3}}$. That means that the three circles with radius $\frac{2}{\sqrt{3}}$ and centers V_1, V_2, and V_4 will intersect in the center of the triangle, and hence fully cover it.