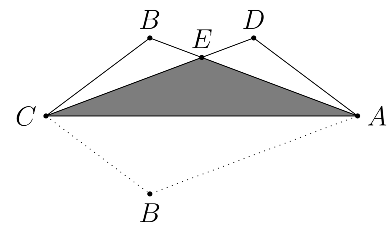
WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

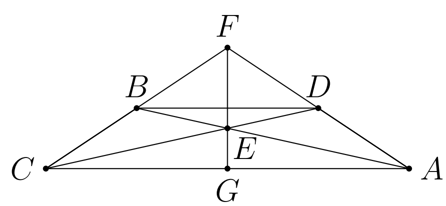
SOLUTIONS TO PROBLEM SET III (2021-2022)

1. A parallelogram $ABCD$ has sides $AB = 7$ and $BC = 3$. When the parallelogram is folded along its long diagonal $\overline{AC}$, the sides $\overline{AB}$ and $\overline{CD}$ cross at point E, and the area of $\triangle ACE$ is the same as the sum of the areas of $\triangle ADE$ and $\triangle CBE$. Find the length of the diagonal $\overline{AC}$.



SOLUTION.

Extend segments $\overline{CB}$ and $\overline{AD}$ so that they meet at point F. Because $\triangle ABC$ is congruent to $\triangle CDA$, $\triangle AFC$ is isosceles. Let G be the midpoint $\overline{AC}$ so that $\overline{FG}$ passes through E. Because $\triangle ADE$ and $\triangle CBE$ are congruent, they each must have area that is half that of $\triangle ACE$. From this it follows that $AE = 2BE$. Because $\overline{AC} \parallel \overline{BD}$, it follows that $\triangle AEC$ is similar to $\triangle BED$. Thus, $AC = 2BD$, and $\overline{BD}$ is the midline of $\triangle AFC$, from which it follows that $\overline{AB}$, $\overline{CD}$, and $\overline{FG}$ are the three medians of $\triangle AFC$ and $CF = AF = 2AD = 6$. Also, the centroid E of $\triangle AFC$ divides the medians in the ratio $2 : 1$, if follows that $AE = \frac{2}{3} \cdot 7$ and $FG = 3EG$.



Let $x = AG$ and $h = EG$. Applying the Pythagorean Theorem to $\triangle AEG$ gives $x^2 + h^2 = \left(\frac{2}{3} \cdot 7\right)^2$, and applying it to $\triangle AFG$ gives $x^2 + (3h)^2 = 6^2$. Multiplying the first of these two equations by 9 and subtracting the second yields $x^2 = 80$, so $x = 2\sqrt{5}$, from which the requested $AC = 2x = 4\sqrt{5}$.

Alternatively, one can use the Law of Cosines. Applying it to $\triangle AFB$ shows $3^2 + 6^2 - 2 \cdot 3 \cdot 6 \cdot \cos F = 7^2$ from which $\cos F = -\frac{1}{9}$. Then applying it to $\triangle AFC$ shows $AC^2 = 6^2 + 6^2 - 2 \cdot 6 \cdot 6 \cdot \cos F = 80$, and $AC = 4\sqrt{5}$, as above.

2. Find the greatest integer n for which $n + 2021$ divides n^{2022}.

SOLUTION. Let $k = n + 2021$, then we need k to divide $(k-2021)^{2022}$. Let us write $(k-2021)^{2022}$ as a product of 2022 copies of $(k - 2021)$ and expand all the brackets. When we do that, all the terms except $(-2021)^{2022} = 2021^{2022}$ will have some power of k in it as a multiplier. All of these terms are divisible by k, and, hence, k divides $(k - 2021)^{2022}$ (the sum of all terms) if and only if k divides 2021^{2022}. The greatest divisor of 2021^{2022} is itself, so the greatest possible value of $k = n + 2021$ is 2021^{2022}, which means that the greatest possible value of n is $2021^{2022} - 2021$.

3. For which positive integer k will the expression $\frac{1776^k + 2022^k}{k!}$ take its greatest possible value? ($k! = k \cdot (k-1) \cdots 2 \cdot 1$ is k factorial.)

SOLUTION. Let $A_k = \frac{1776^k}{k!}$ and $B_k = \frac{2022^k}{k!}$. Then the number we need to maximize is $C_k = A_k + B_k$. Notice that C_k clearly increases when k increases from 1 to 1776 and decreases after $k = 2021$. Indeed, for $1 < k \le 1776$, $A_k/A_{k-1} = 1776/k \ge 1$, and therefore $A_k \ge A_{k-1}$, while $B_k/B_{k-1} = 2022/k > 1$, and therefore $B_k > B_{k-1}$. For $k > 2021$, $A_k/A_{k-1} = 1776/k < 1$ and

$B_k/B_{k-1} = 2022/k \le 1$, which similarly implies that C_k decreases. Hence, the maximizing value of k is somewhere between 1776 and 2022.

Next, notice that C_k actually continues to increase between $k = 1776$ and $k = 2021$. Indeed, for $1776 \le k \le 2021$,

$$A_k - A_{k-1} = \frac{1776^{k-1}}{(k-1)!}\left(\frac{1776}{k} - 1\right) \ge \frac{1776^{k-1}}{(k-1)!}\left(\frac{1776}{2021} - 1\right) = -\frac{1776^{k-1}}{(k-1)!}\frac{245}{2021}$$

and

$$B_k - B_{k-1} = \frac{2022^{k-1}}{(k-1)!}\left(\frac{1776}{k} - 1\right) \ge \frac{2022^{k-1}}{(k-1)!}\left(\frac{2022}{2021} - 1\right) = \frac{2022^{k-1}}{(k-1)!}\frac{1}{2021}.$$

Therefore, for $1776 \le k \le 2021$,

$$C_k - C_{k-1} = (A_k - A_{k-1}) + (B_k - B_{k-1}) \ge -\frac{1776^{k-1}}{(k-1)!}\frac{245}{2021} + \frac{2022^{k-1}}{(k-1)!}\frac{1}{2021}.$$

To show that $C_k - C_{k-1} > 0$ we need to show that

$$\frac{2022^{k-1}}{(k-1)!}\frac{1}{2021} > \frac{1776^{k-1}}{(k-1)!}\frac{245}{2021},$$

which can be simplified to

$$\left(\frac{2022}{1776}\right)^{k-1} > 245.$$

We need this inequality for $1776 \le k \le 2021$, so it is enough to prove that

$$\left(\frac{2022}{1776}\right)^{1775} > 245.$$

Note that $\frac{2022}{1776} \ge \frac{2000}{1800} = \frac{9}{8}$. We claim that $(9/8)^8 > 2$. This could be checked with a calculator, but we can also show it by expanding the product

$$(9/8)^8 = (1 + 1/8)(1 + 1/8)\cdots(1 + 1/8) > 1 + 1/8 + 1/8 + \cdots + 1/8 = 2.$$

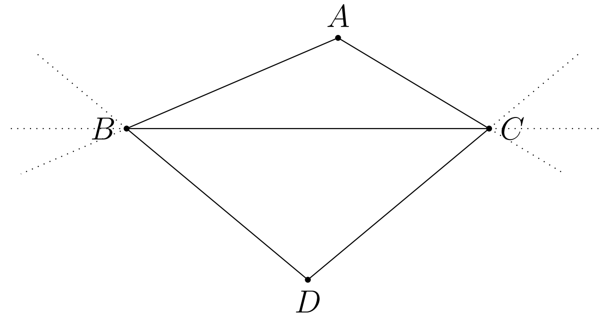
This leads to

$$245 < 256 = 2^8 < (9/8)^{8\cdot 8} \le \left(\frac{2022}{1776}\right)^{64} < \left(\frac{2022}{1776}\right)^{1775}.$$

Altogether, we see that $\frac{1776^k + 2022^k}{k!}$ increases until $k = 2021$ and decreases after $k = 2021$, which means that the expression takes its greatest possible value for $k = 2021$.

4. Keisha drew 100 lines on the plane, with no three lines passing through the same point. These lines divide up the plane into various parts, and some of these parts are triangle-shaped. Jay claims that he can draw another line that does not pass through any of the existing intersection points, and it intersects at least 60 triangle-shaped parts. Show that Jay's claim cannot be true.

SOLUTION. First, notice that two triangular parts cannot be adjacent. Indeed, suppose two parts, $\triangle ABC$ and $\triangle BCD$ are adjacent at the side BC. Then the lines BC, BA, and BD all pass through point B; since we are told that no three lines (drawn by Keisha) pass through the same point, the lines BC and BD coincide. Similarly, the lines AC and AD must coincide as well, and we now have two lines passing through the points C and D.



Now let us disprove Jay's claim. Let us call the line that he draws j. The line j meets each line drawn by Keisha no more than once; therefore, j has at most 100 intersection points with other lines. At each intersection point, j passes from one part of the plane into another; overall, j intersects no more than 101 parts of the plane (two of them infinite). But, if 60 of these 101 parts were triangles, some of them would necessarily be adjacent, contradicting the previous claim. (In fact, there are no more than 99 finite parts, and out of 99 parts, there can be at most 50 triangles).

5. Samir has 1000 cards, each labeled with a different three digit sequence from $(0,0,0)$ to $(9,9,9)$. He also has 100 boxes that are labeled with different two digit sequences from $(0,0)$ to $(9,9)$. He wants to place each of the cards into a box so that the label of the box is the same as the label of the card with one digit deleted. (So the card $(1,2,3)$ can be placed in any of the boxes $(1,2)$, $(2,3)$ and $(1,3)$, but the card $(0,0,0)$ can only be placed in box $(0,0)$.) Show that Samir can place the cards into 50 of the boxes, but he cannot place them into fewer than 50.

SOLUTION. There are $5^2 = 25$ boxes where both digits on the label are even, and there are 25 boxes where both digits on the label are odd. Since each card has three digits on its label, at least two of them will be even or at least two of them will be odd. Hence we can delete one of the three digits so that the other two are both even or both odd. This means that each of the 1000 card can be placed in one of the 50 boxes listed above.

We now show that Samir cannot place the cards in less than 50 boxes. Consider an arrangement of cards into the boxes, and choose the digit a for which the number of empty boxes with labels of the form $(a,*)$ is the maximal. Suppose that there are k such empty boxes, and these are (a,a_1), (a,a_2),..., (a,a_k) where $a_1,a_2,\ldots,a_k$ are different digits. Let $b_1,b_2,\ldots,b_{10-k}$ denote the other $10-k$ digits (that are not one of the a_js).

First note that any card with the label (a,a_i,a_j) was not placed into the boxes (a,a_i) or (a,a_j) (since those are empty by assumption), so it must have been placed in the box with the label (a_i,a_j). Hence, all the boxes with labels (a_i,a_j) (where i and j are between 1 and k) are nonempty. There are $k\cdot k = k^2$ such boxes.

We also note that there could be at most k boxes with labels of the form $(b_j,*)$ that are empty, since a was the digit with the greatest number of empty boxes with labels $(a,*)$ (and that number was k). Hence, for each $1 \le j \le 10-k$ we must have at least $10-k$ nonempty boxes with labels of the form $(b_j,*)$, and this gives at least an additional $(10-k)^2$ extra nonempty boxes. (These are all different from each other for different values of j, and also different from the boxes with labels of the form (a_i,a_j).)

This means that for any arrangements of the cards, we must have at least $k^2+(10-k)^2$ nonempty boxes. The final step is to show that this expression is at least 50. We can do that by checking each possible value of k up to 10 and noting that the expression is least for $k = 5$ when it is exactly 50.

Another way to prove the last step is to note that for any two numbers a and b, we have

$$a^2 + b^2 \geq \tfrac{1}{2}(a+b)^2 = \tfrac{1}{2}a^2 + ab + \tfrac{1}{2}b^2,$$

because rearranging the inequality gives $\frac{1}{2}a^2 - ab + \frac{1}{2}b^2 = \frac{1}{2}(a-b)^2 \geq 0$. Using the inequality for $a = k$ and $b = 10 - k$ gives $k^2 + (10-k)^2 \geq \frac{1}{2} \cdot 10^2 = 50$, finishing the solution.