

WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH
PROBLEM SET IV (2021-2022) **January 2022**

1. We put a green and a yellow marble in a bag. We choose one of them uniformly at random, and return it into the bag with an additional marble that has the same color as the chosen one. We repeat this procedure five more times, after which we have 8 marbles in the bag, some of them green and some of them yellow. What is the probability that there are four green and four yellow marbles in the bag at this point?
2. Prove that a circle centered at the point $(\sqrt{2}, \sqrt{3})$ cannot contain more than one point for which both coordinates are integers.
3. How many integers between 1 and 10^{20} can be represented in the form $\lfloor x^2 \rfloor + \lfloor x \rfloor$ for some positive real number x ? (Here $\lfloor x \rfloor$ refers to the greatest integer less than or equal to x , see Problem 2 of Problem Set 2.)
4. Two 2021-gons, $P_1P_2 \dots P_{2021}$ and $Q_1Q_2 \dots Q_{2021}$, are inscribed in the same circle. It turns out that their corresponding sides are parallel: $P_1P_2 \parallel Q_1Q_2$, $P_2P_3 \parallel Q_2Q_3$, and so on; finally, $P_{2021}P_1 \parallel Q_{2021}Q_1$. Show that the corresponding sides must have equal length: $|P_1P_2| = |Q_1Q_2|, \dots, |P_{2021}P_1| = |Q_{2021}Q_1|$.
5. There is an invisible virus in one of the cells of a 2022×2022 square grid. John has a scanner that can scan the cells of any 50×50 square inside the grid. He can use the scanner once every minute, and each time he can choose where to scan. If the virus is in the scanned area at the time of a scan, then the virus is caught. However, if the virus is not in the scanned area, then it moves to one of the cells that shares a side with its current cell. (This move happens after each unsuccessful scan, before the next scan is made.) Prove that John can find the virus after finitely many scans, and show how this can be done.

You are invited to submit a solution even if you get just one problem. Please do not write your solutions on this problem page. Remember that solutions require a proof or justification.

Find old and current problems and information about the talent search at: <http://www.math.wisc.edu/talent>

Find an introduction to techniques for solving problems like these at: <https://go.wisc.edu/551pe6>

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