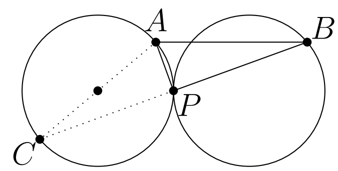
WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET II (2021-2022)

1. Two circles of radius 1 are tangent at the point P. Let A be a point on one of the circles and let B be a point on the other circle such that the angle $\angle APB$ is 90 degrees. Find the length of AB.

SOLUTION. Because the two circles are congruent and tangent at P, the reflection of the plane through point P maps each of the circles into the other. Let C be the reflection of B through P. Then line BP passes through C, as shown. Because $\angle APB = 90°$, it follows that $\angle APC$ is also $90°$ and $\overline{AC}$ is a diameter of the circle containing A. Because $\triangle APB$ and $\triangle APC$ both have right angles and sides $AP = AP$ and $BP = CP$, the triangles are congruent. Therefore, $AB = AC$, which is the diameter of the circles which is 2.



2. For a real number x, we denote by $\lfloor x \rfloor$ the greatest integer less than or equal to x. (This is called the floor function.) For example, $\lfloor 2.3 \rfloor = 2$, $\lfloor -4.2 \rfloor = -5$, and $\lfloor 7 \rfloor = 7$. Find all integer solutions of the following equation:

$$\lfloor x/3 \rfloor = \lfloor x/4 \rfloor + 1.$$

SOLUTION. For any y the number $\lfloor y \rfloor$ is an integer that satisfies

$$\lfloor y \rfloor \leq y < y + 1.$$

This is because $\lfloor y \rfloor$ is the greatest integer less than or equal to y and, hence, $\lfloor y \rfloor + 1$ must be greater than y.

Suppose that x is an integer solution to the equation, and denote $\lfloor x/4 \rfloor$ by m. Then $\lfloor x/3 \rfloor = m + 1$, and we have

$$m \leq x/4 < m + 1, \qquad m + 1 \leq x/3 < m + 2.$$

This gives

$$4m \leq x < 4m + 4 \qquad \text{and} \qquad 3m + 3 \leq x < 3m + 6.$$

Since x, $4m + 4m$, and $3m + 6$ are all integers, we can modify the upper bounds as follows:

$$4m \leq x \leq 4m + 3 \qquad \text{and} \qquad 3m + 3 \leq x \leq 3m + 5. \tag{1}$$

The upper bounds on x must be at least as great as the lower bounds, so we must have $4m \leq 3m+5$ and $3m + 3 \leq 4m + 3$. Solving these inequalities gives $m \leq 5$ and $0 \leq m$. Hence, the possible values of m are $0, 1, 2, 3, 4, 5$. Writing down the inequalities (1) for each of the possible values, we get the possible x values:

$$\begin{array}{rcccc} m=0, & 0 \le x \le 3, & 3 \le x \le 5 & \rightsquigarrow & x=3 \\ m=1, & 4 \le x \le 7, & 6 \le x \le 8, & \rightsquigarrow & x=6,7 \\ m=2, & 8 \le x \le 11, & 9 \le x \le 11, & \rightsquigarrow & x=9,10,11 \\ m=3, & 12 \le x \le 15, & 12 \le x \le 14 & \rightsquigarrow & x=12,13,14 \\ m=4, & 16 \le x \le 19, & 15 \le x \le 17, & \rightsquigarrow & x=16,17 \\ m=5, & 20 \le x \le 23, & 18 \le x \le 20, & \rightsquigarrow & x=20. \end{array}$$

All these x values satisfy the original equation, and these are the only solutions: $3, 6, 7, 9, 10, 11, 12, 13, 14, 16, 17, 20$.

3. Let $p(n) = n^3 - 12n^2 + 32n + 3$. Note that $p(4)$ is a prime number. Find all the integers n such that $p(n)$ is a (positive) prime number.

SOLUTION. Note that $p(4) = 3$. We have

$$p(n) - 3 = n^3 - 12n^2 + 32n = n(n^2 - 12n + 32) = n(n-4)(n-8).$$

This shows that $p(n) - 3 = 0$ exactly for $n = 0, 4$, and 8. In these cases the polynomial is equal to 3; hence, it is a positive prime.

We will show that these are the only values of n that produce a prime number. Indeed, note that for an integer n one of the integers n, $n-4$, and $n-8$ will be divisible by three. (This follows by checking the three possible remainders that n can give when divided by 3.) This implies that $p(n) - 3 = n(n-4)(n-8)$ is always divisible by three, which means that $p(n)$ must be divisible by three as well. Hence, if $p(n)$ is a positive prime, then it must be equal to 3, and we have seen that this is only possible when n is equal to 0, 4, or 8.

4. We write the numbers $1, 2, 2^2, 2^3, \ldots, 2^{2021}$ on a (very large) blackboard. We are allowed to make the following changes to these numbers: we can erase (any) two numbers on the board, and then write the absolute value of their difference on the board. We repeat this process 2021 times, after which we have only one number remaining on the board. Show that the last number has to be an odd integer between 1 and $2^{2021} - 1$, and, conversely, that any odd integer between 1 and $2^{2021} - 1$ can be obtained by this procedure.

SOLUTION. In the beginning, there is exactly one odd number written on the board, namely, $2^0 = 1$. Since the difference of two even numbers is even, while the difference of an odd number and an even number is odd, after each step, there is still going to be exactly one odd number; therefore, at the end, the remaining number must be odd.

Also, note that all numbers in the beginning do not exceed 2^{2021}. If two (non-negative) numbers are less or equal than 2^{2021}, then so is their difference. Therefore, after each step, all the numbers written on the board do not exceed 2^{2021}. In particular, the number remaining at the end must be less or equal than 2^{2021}. Since it is odd and non-negative, it is in fact an odd integer between 1 and $2^{2021} - 1$, as required

Let us now show that any odd integer between 1 and $2^{2021} - 1$ can be obtained. More generally, let us prove that, for any given $N > 0$, we can obtain any odd integer M between 1 and $2^N - 1$ starting from the powers of 2

$$2^0 = 1, 2^1 = 2, \ldots, 2^N.$$

The proof proceeds by induction in N. If $N = 1$, the only possible value of M is 1, which can be obtained as the difference

$$2^1 - 2^0 = 1.$$

For the induction step, we suppose that the claim holds for $N = k$ for some positive integer k, and we have to prove the claim for $N = k + 1$. Thus, suppose M is an odd integer between 1 and $2^{k+1} - 1$. Let us describe how to obtain it. We consider two cases:

Case 1. $M < 2^k$. In this case, our first action is to replace 2^{k+1} and 2^k (the two largest numbers on the board) by their difference:

$$2^{k+1} - 2^k = 2^k.$$

Now the board has the numbers

$$2^0 = 1, 2^1 = 2, \ldots, 2^k,$$

and we know by the induction hypothesis that it is possible to obtain M from them.

Case 2. $M > 2^k$. Let us put $M' = 2^{k+1} - M$. We see that M' is an odd integer between 1 and $2^k - 1$; therefore, by the induction hypothesis, it can be obtained starting from the powers of 2 between 1 and 2^k. Let us do so, keeping the greatest power (2^{k+1}) untouched. Afterwards, the board has only two numbers:

$$M', 2^{k+1}.$$

As the last action, we replace the two numbers by their difference

$$2^{k+1} - M' = M,$$

obtaining M, as required.

Remark. As an aside, note that in the algorithm described above, we only subtract neighboring pairs of integers. (Here we assume that, whenever we erase a pair of neighbors, we write their difference in their place.) In fact, if we write the powers of 2 in the descending order

$$2^N, 2^{N-1}, \ldots, 1$$

we can interpret the process as a placement of parentheses in the expression

$$2^N - 2^{N-1} - \cdots - 1.$$

For instance, here are the placements of parentheses for $N = 3$ giving all possible odd numbers between 1 and $2^3 - 1 = 7$:

$$((8-4)-2)-1 = 1$$
$$(8-4)-(2-1) = 3$$
$$8-(4-(2-1)) = 5$$
$$8-((4-2)-1) = 7.$$

5. There are 50 girls and 50 boys standing in line in some order. We know that there is exactly one stretch of 30 children next to each other with an equal number of boys and girls. Show that there is also a stretch of 70 children in a row with an equal number of girls and boys.

SOLUTION. First of all, define the sequence a_1, a_2, $\ldots$, a_{100} of numbers ± 1 as $a_k = +1$ if kth child in the line is a girl, and $a_k = -1$ if kth child in the line is a boy. We know that exists exactly one $m \leq 71$ such that $a_m + a_{m+1} + \ldots + a_{m+29} = 0$, and need to show that exists $\ell \leq 31$ such that $a_\ell + a_{\ell+1} + \ldots + a_{\ell+69} = 0$.

Main tool of our solution (and solutions for many similar problems) is the following fact:

Fact 1. (The Discrete Intermediate Value Theorem): Suppose that we have a sequence of integers where the difference of any two adjacent terms is -1, 0 or 1. If the sequence contains both a positive and a negative integer, then it must contain zero as well.

Proof of Fact 1: Suppose such a sequence contains at least one positive integer and one negative integer. If a positive integer m is adjacent to a negative integer n, then their difference is either $m - n > 1 - (-1) = 2$ or $n - m < -1 - 1 = -2$. In either case, the difference is not one of -1, 0, or 1. Thus, there are no positive terms adjacent to negative terms, so there must be terms equal to 0 between the positive term and the negative term.

An application of Fact 1 is the following statement:

Fact 2. Suppose we have a sequence whose terms are all $+1$ and -1 in some order. Let $n \geq 2$ be an even positive integer such that there are n adjacent terms in the sequence with a positive sum and another n adjacent terms with a negative sum. Then there must be n adjacent terms somewhere in the sequence with a sum equal to 0.

Proof of Fact 2: Denote the numbers in the sequence in order by $b_1, b_2, b_3, \ldots$. By our assumption, there are integers k and ℓ so that $b_k + b_{k+1} + \ldots + b_{k+n-1} < 0$ and $b_\ell + b_{\ell+1} + \ldots + b_{\ell+n-1} > 0$. We will show that there exists m between k and ℓ, such that $b_m + b_{m+1} + \ldots + b_{m+n-1} = 0$. Consider a new sequence which at position x has the number $c_x = \frac{1}{2}(b_x + b_{x+1} + \ldots + b_{x+n-1})$. (This is half the sum of the n numbers in the original sequence starting at position x.) Note that the new sequence has integer values: because n is even and $b_y = \pm 1$, the sum $b_x + b_{x+1} + \ldots + b_{x+n-1}$ is always even. Note also that $c_x - c_{x+1} = \frac{1}{2}(b_x - b_{x+n})$ which can only be -1, 0 or 1. (Since both b_x and b_{x+n} are ± 1.) This means that we can apply Fact 1 for the sequence $c_1, c_2, \ldots$: since $c_k < 0$ and $c_\ell > 0$ we must have an m between k and ℓ so that $c_m = \frac{1}{2}(b_m + b_{m+1} + \ldots + b_{m+n-1}) = 0$, proving Fact 2.

We will use this Fact 2 for $n = 30$ and $n = 70$. Less formally speaking, Fact 2 says that if we have negative sum of 30 (resp. 70) consecutive elements somewhere and positive sum of 30 (resp. 70) consecutive elements somewhere else in a sequence of ± 1, then we have zero sum of 30 (resp. 70) consecutive elements somewhere between them.

Let's get back to our original problem: we have a sequence $a_1, a_2, \ldots, a_{100}$ with fifty $+1$ and fifty -1 in some order. We know that there is exactly one stretch of 30 numbers that add up to 0, we have to show that there is a stretch of 70 numbers that sum to zero as well.

Below we present two ways to finish solution.

Solution 1: Let's arrange our numbers $a_1, \ldots, a_{100}$ around the circle clockwise (so now a_{100} and a_1 are neighbours). For convenience, define $a_n = a_{n-100}$ for $n > 100$, and look at all possible sums $a_m + \ldots + a_{m+29}$. We have exactly 100 such sums on the circle. Every element of the sequence $a_1, \ldots, a_{100}$ participates in exactly 30 of the sums, so if we add all 100 of the sums, we just get $30(a_1 + \cdots + a_{100}) = 0$. Not all of these sums are equal to zero (e.g. at most one of $a_1 + \cdots + a_{30}$ and $a_2 + \cdots + a_{31}$ can be zero by our assumption), which means that there must be at least one positive sum and at least one negative sum among the 100 sums. (Otherwise they cannot add up to 0.)

Let $a_k + \ldots + a_{k+29} > 0$ and $a_\ell + \ldots + a_{\ell+29} < 0$. We may assume that $1 \le k < \ell \le 100$. (Otherwise the solution goes similarly). Now, on the circle, we can apply Lemma 2 twice, from k to ℓ clockwise and from ℓ to k clockwise. So, we found two indexes m_1 and m_2 (m_1 between k and ℓ, m_2 between ℓ and $k+100$), such that $a_{m_i} + \ldots + a_{m_i+29} = 0$. But we know that exists exactly one index m between 1 and 71 such that $a_m + \ldots + a_{m+29} = 0$. It means, $m_1 = m$, and $m_2 > 71$, i.e. the sum $a_{m_2} + \ldots + a_{m_2+29}$ includes both a_{100} and a_1.

Now look at the complementary sum of other 70 elements of $a_{m_2} + \ldots + a_{m_2+29}$. We have

$$a_{m_2-70} + \ldots a_{m_2-1} = (a_1 + \ldots + a_{100}) - (a_{m_2} + \ldots + a_{m_2+29}) = 0.$$

Because the sum $a_{m_2} + \ldots + a_{m_2+29}$ includes both a_{100} and a_1, the complementary sum $a_{m_2-70} + \ldots a_{m_2-1}$ includes an actual stretch of 70 numbers next to each other in the original sequence that sum to zero, proving the statement.

Solution 2: Consider all sums of 70 elements next to each other in our sequence: $a_m + \ldots + a_{m+69}$ for all m from 1 to 31. If one of them equal 0, we are done. If there are two with different signs, we can apply Fact 2 for them and find a sum that is equal to zero. That means that they are all positive or all negative. Let's assume that they are all positive. (The other case is similar.) Hence, the complementary sums $a_1 + \ldots + a_{30}$ and $a_{71} + \ldots + a_{100}$ are both negative ($a_1 + \ldots + a_{100} = 0$).

Case 1. There is a k between 1 and 71 such that $a_k + \ldots + a_{k+29} > 0$. Then Fact 2 gives us 2 different m_1 and m_2 (m_1 between 1 and k, m_2 between k and 71), such that $a_{m_i} + \ldots + a_{m_i+29} = 0$, which contradicts with statement of the problem.

Case 2. For all k between 1 and 71 we have $a_k + \ldots + a_{k+29} \le 0$. To obtain a contradiction we carefully check sign of sums of ten consecutive elements. Let

$$\begin{aligned} A_1 &= a_1 + a_2 + \ldots + a_{10} \\ A_2 &= a_{11} + a_{12} + \ldots + a_{20} \\ &\ldots \\ A_9 &= a_{81} + a_{82} + \ldots + a_{90}. \end{aligned}$$

In this notation we have

$$\begin{array}{llll} A_1 + A_2 + \ldots + A_7 > 0, & A_1 + A_2 + A_3 \le 0, & A_4 + A_5 + A_6 \le 0 & \Longrightarrow A_7 > 0; \\ A_2 + A_3 + \ldots + A_8 > 0, & A_2 + A_3 + A_4 \le 0, & A_5 + A_6 + A_7 \le 0 & \Longrightarrow A_8 > 0; \\ A_3 + A_4 + \ldots + A_9 > 0, & A_3 + A_4 + A_5 \le 0, & A_6 + A_7 + A_8 \le 0 & \Longrightarrow A_9 > 0. \end{array}$$

So, we have $A_7 + A_8 + A_9 = a_{61} + \cdots + a_{90} > 0$, which contradicts our assumption, completing the solution.