WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET I (2021-2022)

1. We have four identically looking gold chains with weights 1, a, a^2, and a^3, where $a > 1$ is an unknown value. We also have a balance scale with two pans that can be used to compare weights. When used, the scale only shows which side is heavier, or that the two sides weight the same; it does not show the actual weights. Find a way to identify the heaviest of the four chains by using two comparisons. (You can decide which and how many chains you put in the pans.)

SOLUTION. For the first weighting we can place two chains in each pan. The weights of the chains will then divided as $(1, a)$ vs (a^2, a^3), $(1, a^2)$ vs (a, a^3), or $(1, a^3)$ vs (a, a^2). Note that the chain with weight a^3 is always in the pan with the heavier weight because

$$a^3 + a^2 > a + 1, \qquad a^3 + a > a^2 + 1, \qquad \text{and} \qquad a^3 + 1 > a^2 + a.$$

The first two inequalities follow from $1 < a < a^2 < a^3$. To show the last one we can check that

$$a^3 - a^2 + 1 - a = a^2(a - 1) - (a - 1) = (a^2 - 1)(a - 1),$$

which is always positive because $1 < a$ and $1 < a^2$.

This means that after the first comparison, we could identify two chains so that one of them must be the one weighing a^3. Using an additional weighing to compare these two chains to each other (and choosing the heavier one) we will have identified the the chain weighing a^3.

2. Each square of a 100×100 board is painted in one of 20 available colors. Call a square *lonely* if its color is different from all other squares in its row and in its column. What is the maximal possible number of lonely squares on the board?

SOLUTION. The answer is 1900.

Each lonely square in a row is the only square of a certain color in this row. If we had more than 19 lonely squares in a row, then we would need to use all 20 available colors in that row, which would mean that at least one of the colors was used multiple times (and the corresponding color could not be the color of a lonely square). Hence, there can be no more than 19 lonely squares in each row. Since there are 100 rows, the total number of lonely squares cannot exceed $19 \times 100 = 1900$.

Since it is impossible to have more than 1900 lonely squares, it is sufficient to show a way to paint the board with 1900 lonely squares. Paint 1900 of the squares in 19 of the colors so that in each row and in each column there is exactly one square of each of the 19 colors; this way, all of these 1900 squares will be lonely. The rest of the board can be painted in the 20th color, and none of those squares will be lonely. This can be accomplished as follows: Given a square in the i-th row and j-th column, paint it the first color if $i = j$ (that is, if the square is on the main diagonal), the second color if $i = j + 1$ or $i = j - 99$ (which happens if the square is directly below the main diagonal or in the top right corner), the third color if $i = j + 2$ or $i = j - 98$ (which happens if the square is two squares below the main diagonal or 98 squares above it) and so on. The square will be painted the 19th color if either $i = j + 18$ or $i = j - 82$. This way for each of the first 19 colors we painted exactly 100 squares, and no two squares with the same color are in the same row or column. The remaining squares are painted the 20th color.

1								19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2
2	1								19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3
3	2	1								19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4
4	3	2	1								19	18	17	16	15	14	13	12	11	10	9	8	7	6	5
5	4	3	2	1								19	18	17	16	15	14	13	12	11	10	9	8	7	6

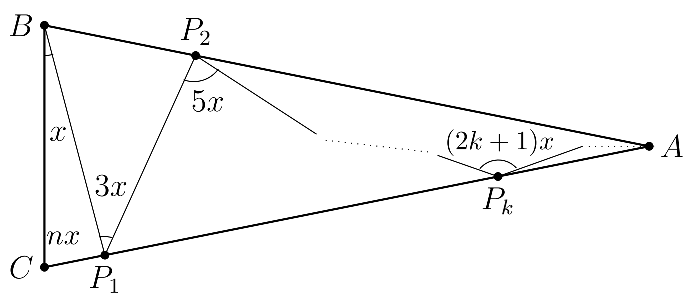
3. In an acute isosceles triangle one of the angles is n times as large as another one, where $n > 1$ is an integer. Show that the triangle can be cut into n smaller triangles with $n-1$ straight cuts so that all the resulting smaller triangles are isosceles.

SOLUTION. An isosceles triangle has two equal base angles and a vertex angle, and these add up to $180°$. If the vertex angle is at least $n \geq 2$ times a base angle, then it has to be at least as large as the sum of the other two angles. But then it hast to be at least $180°/2 = 90°$, which means that the triangle cannot be acute.

So, it must be that the base angle of the isosceles triangle is n times the vertex angle. Let the vertices of the triangle be A, B and C, with $\angle BAC = x$ and $\angle ABC = \angle ACB = nx$. Then $\angle BAC + \angle ABC + \angle ACB = (2n+1)x = 180°$.

The cuts can be made in the following way. Choose a point P_1 on the side $\overline{AC}$ so that $\angle P_1BC = x$. Then $\angle BP_1C = 180° - x - nx = (2n+1)x - (n+1)x = nx = \angle BCP_1$, and the triangle $\triangle BCP_1$ is isosceles. The angle $\angle BP_1A$ is equal to $180° - \angle BP_1C = (2n+1)x - nx = (n+1)x$, and $\angle P_1BA = nx - x = (n-1)x$. Hence, the angles of $\triangle BP_1A$ are $x, (n+1)x, (n-1)x$.

If $n = 2$, then the triangle $\triangle BP_1A$ is also isosceles (since $x = (n-1)x$), and we are done. If $n > 2$, then choose a point P_2 on the side $\overline{AB}$ so that $\angle BP_1P_2 = 3x$. Notice that $\angle P_1BP_2 = nx - x = (n-1)x$, and $\angle BP_2P_1 = 180° - (n-1)x - 3x = (2n+1)x - (n-1)x - 3x = (n-1)x$. Therefore, $\triangle BP_1P_2$ is isosceles. Once again, if $n = 3$, then we are finished. If $n > 3$, choose a point P_3 on P_1A so that $\angle P_1P_2P_3 = 5x$. Check that then $\angle P_2P_1P_3 = \angle P_2P_3P_1 = (n-2)x$ and $\triangle P_1P_2P_3$ is isosceles.



If $n > 4$, we can reiterate the same procedure choosing points P_k for $k = 4, 5, ..., n-1$. After the point P_k is chosen on one of the sides, if $k < n-1$, we choose the next point P_{k+1} on $P_{k-1}A$ so that $\angle P_{k-1}P_kP_{k+1} = (2k+1)x$, check that $\angle P_kP_{k-1}P_{k+1} = \angle P_kP_{k+1}P_{k-1} = (n-k)x$, and conclude that $\triangle P_{k-1}P_kP_{k+1}$ is isosceles. If $k = n-1$, then $\triangle P_kP_{k+1}A$ is isosceles, and we are finished.

4. Find all integers d for which there are infinitely many integer solutions (x, y) of the equation

$$x^2 + x + 2021 = y^2 + 3y + d.$$

SOLUTION. Multiply both sides of the equation by 4, and complete the square on both sides:

$$4x^2 + 4x + 8084 = 4y^2 + 12y + 4d,$$
$$(2x + 1)^2 + 8083 = (2y + 3)^2 + 4d - 9.$$

Rearranging the two sides, and then using the product identity for differences of squares:

$$8092 - 4d = (2y + 3)^2 - (2x + 1)^2 = (2y + 3 + 2x + 1)(2y + 3 - (2x + 1))$$
$$8092 - 4d = (2x + 2y + 4)(2y - 2x + 2).$$

We show that this equation has infinitely many integer solutions (x, y) when $d = 2023$, and finitely many solutions otherwise.

Indeed, if $d = 2023$, then $8092 - 4d = 0$, and any pair (x, y) with $y = x - 1$ will be a solution since in this case $2y - 2x + 2 = 0$ as well.

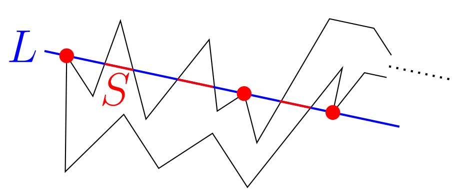
If $d \neq 2023$, then $8092-4d$ is a nonzero integer. Since $(2x+2y+4)(2y-2x+2) = 8092-4d$, we have that $2x+2y+4$ and $2y-2x+2$ are both integers that divide $8092-4d$. Since $8092-4d \neq 0$, we have finitely many pairs of integers with a product equaling $8092 - 4d$. For any particular choice of a and b with $ab = 8092 - 4d$, there is at most one integer solution of

$$2x + 2y + 4 = a, \quad 2y - 2x + 2 = b$$

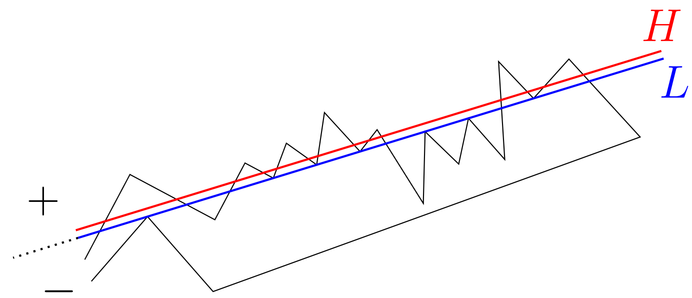
since solving this system of equations gives $x = \frac{a-b-2}{4}$ and $y = \frac{a+b-6}{4}$, and these may or may not be integers. Hence, if $8092 - 4d \neq 0$, there are finitely many integer solutions (x, y).

5. Let P be a polygon and L be a line such that L intersects P in exactly 2021 points. Show that there is another line that intersects P in more than 2021 points. (A polygon is a finite number of straight line segments that connect end-to-end to form a closed path that does not intersect itself.)

SOLUTION. Consider the set S of points on L that are either on P or in the interior of P. Then S consists of some segments of positive length and some isolated points. Each of the segments of positive length intersect P at 2 points, and each of the isolated points must be vertices of the polygon P. Thus, for there to be 2021 intersections points, S must contain an odd number of isolated points, all of which are vertices of the polygon P.



Designate the half-plane on one side of L as the positive half-plane and the other side as the negative half-plane. At each of the isolated points of S, two sides of P meet, and both of those sides must be on the same side of L, so they are both in the positive half-plane or the negative half-plane. Designate an isolated point of S as positive if the two sides that meet at that point are in the positive half-plane, and otherwise designate the isolated point as negative. Because there are an odd number of these isolated points in S, the number of positive isolated points cannot equal the number of negative isolated points. Assume without loss of generality that there are more positive isolated points than negative isolated points.



Let α be the minimum positive distance that a vertex of P is from L. Let H be the line parallel to L in the positive half-plane that is a distance $\frac{\alpha}{2}$ from L. Then H will intersect P in a finite number of points that is more than 2021 points. Indeed, if L intersects P at a point that is not an isolated point of S, then H will also intersect P on the same side of P because H is not far enough from L to reach the end of that side. If L intersects P at a positive isolated point of P, then H will intersect both sides of P that meet at that vertex, and since there are more positive isolated points than negative isolated points in S, this shows that H will intersect P in more points than L did.