WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

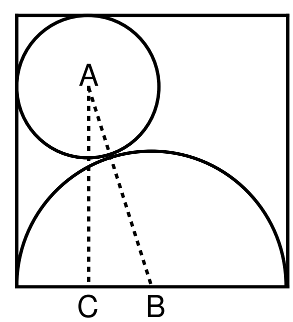
SOLUTIONS TO PROBLEM SET I (2005-2006)

1. Let S be some set of positive integers and let T be the set of all numbers of the form $x + y$, where x and y are in S, allowing $x = y$. Suppose that T is the complement of S in the set of positive integers. (In other words, T is exactly the set of positive integers that do not lie in S.) Show that the number $1,000,000$ is a member of T.

SOLUTION. We will prove that S must be exactly the set of all odd positive integers, and so T is the set of all even positive integers. If this is false, let n be the smallest number that is in the "wrong" set. In other words, if n is odd, it is in T and if n is even, it is in S, but all odd numbers smaller than n are in S and all even numbers smaller than n are in T. We work to derive a contradiction.

First, assume that n is odd and so lies in T. We can thus write $n = x + y$ where x and y are in S. Since x and y are positive, we have $x < n$ and $y < n$, and because these numbers lie in S, they must be odd, and hence $n = x + y$ is even, which is a contradiction. Now assume that n is even and so lies in S. Note that $n \geq 2$, so $n - 1$ is positive and odd and smaller than n. Thus $n - 1$ is in S and similarly 1 is in S. Thus $n = 1 + (n - 1)$ is a sum of two members of S, and hence it lies T. This is a contradiction, however, since n is in S. We obtained contradictions both when n is odd and when n is even, and this establishes our assertion. In particular, T is the set of all even positive integers, and so $1,000,000$ lies in T.

2. In the diagram, the length of each side of the square is 2 and the bottom side of the square is the diameter of the semicircle. The circle is tangent to the semicircle and to two sides of the square, as shown. Compute (with proof) the radius of the circle.



SOLUTION. In the diagram, A is the center of the circle and B is the center of the semicircle (so that B is also the midpoint of the bottom side of the square.) Also, $\overline{AC}$ is the perpendicular dropped from A to the bottom side of the square. Let the radius of the circle be r and note that since the radius of the semicircle is 1, we have $AB = 1 + r$. Point A is at distance r from the top of the square, so its distance to the bottom of the square is $2 - r$, and we have $AC = 2 - r$. The distance from A to the left side of the square is r, and thus this is also the distance from C to the lower left corner of the square. (This is because $\overline{AC}$ is parallel to the left side of the square.) It follows that $BC = 1 - r$. We can now apply the Pythagorian theorem in $\triangle ABC$ to deduce that $(2 - r)^2 + (1 - r)^2 = (1 + r)^2$. Thus $(4-4r+r^2)+(1-2r+r^2) = 1+2r+r^2$. This yields $r^2 - 8r + 4 = 0$. Applying the quadratic formula, we get $r = (8 \pm \sqrt{64 - 16})/2$, and simplification yields $r = 4 \pm 2\sqrt{3}$. Since $\sqrt{3} > 1$ and it is obvious that r cannot be as large as 6, we clearly need the minus sign, and thus $r = 4 - 2\sqrt{3}$.

3. One season, each team in a sports league won at least five games against other teams in the league. Prove that some team lost at least five games that season.

SOLUTION. Let n be the number of teams in the league. Since each of the n teams was the winner of at least 5 games within the league, there must have been at least $5n$ intraleague games that did not end in ties. Each of these games had a loser, and so the average number of losses per team is at least $5n/n = 5$. It is not possible that every team had a below-average number of losses, so at least one team had at least five losses.

4. Let a be an odd positive integer and suppose that the equation $x^2 - y^2 = a$ has exactly one solution in positive integers x and y. Prove that either a is a prime number or it is the square of a prime number.

SOLUTION. Consider factorizations $a = uv$, where u and v are integers and $u < v$. We argue that each such factorization determines a positive integer solution to the equation $x^2 - y^2 = a$, and that different factorizations of this form yield different solutions. To find x and y given u and v, set $x = (v + u)/2$ and $y = (v - u)/2$, and observe that x and y are positive integers since u and v are odd and $u < v$. Now $v = x + y$ and $u = x - y$, and thus $x^2 - y^2 = (x - y)(x + y) = uv = a$, and we have a solution, as wanted. Also, since $u = x - y$ and $v = x + y$, we see that no other factorization can yield the same solution. We mention also that every positive integer solution of the equation $x^2 - y^2 = a$ actually does come from a factorization $a = uv$ with $u < v$. To see this, simply take $u = x - y$ and $v = x + y$.

Given that the equation $x^2 - y^2 = a$ has a solution, it is evident that $a > 1$. Since there is only one solution, it must be true that there is only one integer factorization $a = uv$ with $u < v$, and thus the factorization $a = 1a$ is the only such factorization. It follows that if we can write a as the product of two integers exceeding 1, then the factors must be equal. Let p be a prime divisor of a. Since we have the factorization $a = p(a/p)$ and $p > 1$, the only possibilities are $a/p = 1$ or $a/p = p$. In the first case, $a = p$ is prime, and in the second case, $a = p^2$ is the square of a prime.

In the case where a is an odd prime or the square of an odd prime, the equation $x^2 - y^2 = a$ really does have a unique solution in positive integers. The factorization $a = 1a$ yields one solution, and since there is no other factorization $a = uv$ with $u < v$, there can be no other solution.

5. Consider numbers $a_1 < a_2 < a_3 < a_4$ and $b_1 < b_2 < b_3 < b_4$. Suppose also that $a_2 < b_1$ and that $a_4 < b_4$. If $a_1 + a_2 + a_3 + a_4 \geq b_1 + b_2 + b_3 + b_4$, determine the ordering of these eight numbers. In other words, say which is the smallest, the second smallest, the third smallest, *etc.* (Of course, a proof is required here.)

SOLUTION. From the given data, we have $a_1 < a_2 < b_1 < b_2$ and in particular, $a_1 < b_1$ and $a_2 < b_2$. We are also given that $a_4 < b_4$, and so we know that for subscripts i in the set $\{1, 2, 4\}$, we have $a_i < b_i$. But the sum of all four numbers a_i is not smaller than the sum of all four b_i, and we deduce from this that $a_3 > b_3$. Thus $b_3 < a_3 < a_4 < b_4$ and we have $a_1 < a_2 < b_1 < b_2 < b_3 < a_3 < a_4 < b_4$. This is therefore the only possble ordering of the given eight numbers.