WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET V (2020-2021)

1. Jodie chooses at most 7 prime numbers which are greater than 10, computes the sum of their squares, and tells this sum to Emily. Show that Emily can always identify the number of primes Jodie chose. The primes that Jodie chooses do not have to be different, and Emily does not need to identify the actual primes chosen by Jodie.

SOLUTION. We show that if a prime number p is greater than 10, then p^2 has remainder 1 when divided by 8. In other words: p^2 modulo 8 is equal to 1. From this it follows that the sum of the squares computed by Jodie will have the same remainder modulo 8 as the number of terms in the sum. Since the number of terms is at most 7, the remainder of the sum modulo 8 will be exactly equal to the number of terms.

If p is a prime that is larger than 10, then p is odd, and it can be written as $p = 2k+1$ with an integer k. (In fact, this is true for any prime greater than 2.) Then $p^2 = (2k+1)^2 = 4k^2+4k+1 = 4k(k+1)+1$. Note that either k or $k+1$ is even, hence $4k(k+1)$ is divisible by 8. This shows that p^2 has a remainder 1 when divided by 8.

2. Maya and Yolanda take turns writing positive integers that are at most 1000 on a large whiteboard, with Maya going first. A player is not allowed to write the number 13, to write a number which has already appeared on the board, or to write a number which differs by 17 from a number which already appears on the board. The first player who cannot write another number loses. Which player has a winning strategy? (You also have to provide a possible winning strategy.)

SOLUTION. We divide all numbers from 1 to 1000 at 17 arithmetic progressions with common difference 17 (we will name them "rows"):

$$\begin{array}{rrrcr}
1, & 18, & 35, & \dots, & 987; \\
2, & 19, & 36, & \dots, & 988; \\
\dots & \dots & \dots & \dots & \dots \\
12, & 29, & 46, & \dots, & 998; \\
 & 30, & 47, & \dots, & 999; \\
14, & 31, & 48 & \dots, & 1000; \\
 & 15, & 32, & \dots, & 984; \\
 & 16, & 33, & \dots, & 985; \\
 & 17, & 34, & \dots, & 986.
\end{array}$$

Because $1000 = 17 \cdot 58 + 14$, the first 12 rows and the 14th row have length 59 (*long rows*), and other 4 rows (13th, 15th, 16th and 17th) have length 58 (*short rows*). Two numbers differ by exactly 17 if and only if they are adjacent in one of the rows.

We provide a winning strategy for Maya. There are 13 long rows and 4 short rows. By removing the 1st long row, we can divide up the remaining 12 long rows into 6 pairs, and the 4 short rows into two pairs.

Maya writes $1 + 17 \cdot 29$ for her first number, this is the number in the middle position of the first long row. The remaining numbers in the first row can be paired up in a symmetric way: the number $1 + 17 \cdot k$ is paired up with $1 + 17 \cdot (58 - k)$, where $0 \le k \le 58$ and $k \ne 29$.

After her first move, Maya will use the following strategy:

– If Yolanda writes down a number a from the first row, then in the next step Maya will write down the one in the 1st row that is paired up with it.

– If Yolanda writes down the mth number from any other row, then Maya follows by writing down the mth number from the paired row in the next step.

Using these rules Maya will always be able to write a number down since whenever Yolanda writes down a number, there is a number paired with it that has not been used yet. (Note that in our pairings the two numbers never differ by 17.) Moreover, if Yolanda has not written down a number that differs from another one by 17, then this will hold after Maya's subsequent turn as well. If Maya's chosen number x differs from a number y by 17 that has already been chosen, then we have two possibilities:

- $y = 1 + 17 \cdot 29$ (Maya's first chosen number), which means that x is $1 + 17 \cdot 28$ or $1 + 17 \cdot 30$. Then Yolanda's previous number is $1 + 17 \cdot 30$ or $1 + 17 \cdot 28$, which differs from Maya's first chosen number by 17.

- We have $y \neq 1 + 17 \cdot 29$. In this case the respective pairs of both x and y have already been written on the board, and these numbers have to differ by 17 as well (which cannot happen).

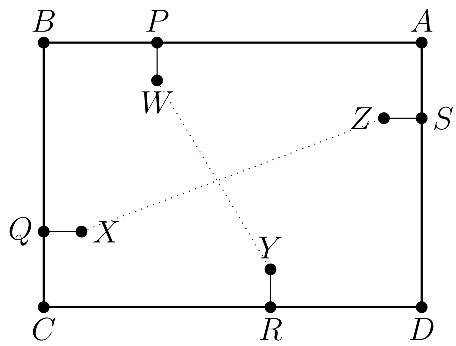
This means that using the strategy Maya can always write a new number satisfying the conditions, and since there are only 999 numbers, at some point Yolanda will run out of possible numbers to write. Hence using this strategy Maya will always win.

3. Find the smallest integer $n \geq 2$ such that any rectangle can be divided into n non-overlapping smaller rectangles (with sides parallel to the original one) in a way that no pair of smaller rectangles has two vertices in common. (You must provide a construction for n, and prove that $n - 1$ is impossible.)

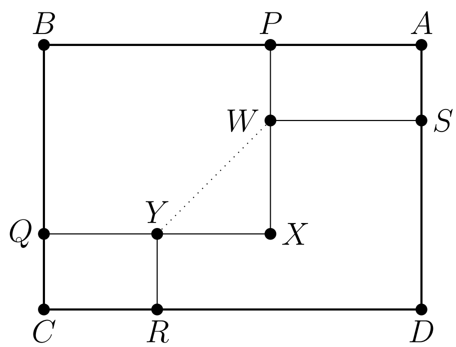
SOLUTION. First note that if any rectangle can be divided into n rectangles in the prescribed way, then by dilations in the directions of the sides of the rectangles it is seen that all rectangles can be divided into n rectangles.

Suppose that n is the least integer greater than 1 such that a rectangle can be divided into n rectangles in the prescribed way, and suppose that such a division of rectangle $ABCD$ contains a rectangle $ABC'D'$. That would mean that rectangle $C'D'CD$ can be divided into $n-1$ rectangles implying that n was not minimum. Thus, for the optimal n there must be at least one rectangle for each vertex of $ABCD$, and n must be at least 4.

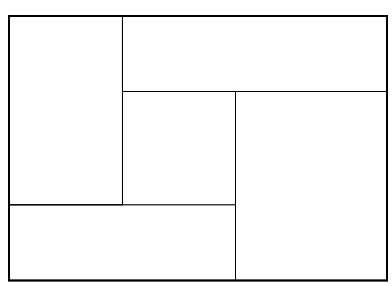
Suppose then that $n = 4$. The 4 rectangles in a division of $ABCD$ must include the four vertices A, B, C, and D, with one vertex in each rectangle of the division. Thus, there must be points P, Q, R, and S on sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, respectively, and four points W, X, Y, and Z in the interior of $ABCD$ such that the four rectangles of the division are $APWS$, $BQXP$, $CRYQ$, and $DSZR$. If $W = X$, $X = Y$, $Y = Z$, or $Z = W$, then two rectangles of the division would share two vertices, and that is not allowed. Also, if $W = Y$, then the rectangle containing A must touch the rectangle containing C, requiring all four rectangles to meet at $W = Y$, which is also not allowed. Similarly, $X = Z$ is not allowed. Thus, W, X, Y, and Z are four distinct points.



If Y is in $\overline{WS}$, then it must be that $X = W$, which is not possible. Similarly, Y cannot be in $\overline{WP}$. Thus, the interior of segment $\overline{WY}$ is not covered at all by $APWS$ or $CRYQ$, and, similarly, the interior of segment $\overline{XZ}$ is not covered at all by $BQXP$ or $DSZR$. It follows that the intersection of $\overline{WY}$ and $\overline{XZ}$ is not covered, so it must be empty. Then, because lines WY and XZ cannot intersect at a point on both segments, either $\overline{WY}$ lies completely on one side of line XZ or $\overline{XZ}$ lies completely on one side of line WY. Assume, without loss of generality, that $\overline{XZ}$ and B lie on opposite sides of line WY. Then $\overline{WY}$ cannot intersect $DSZR$, and it follows that segment $\overline{WY}$ is contained completely inside either $BQXP$. But if $\overline{WY}$ is covered by $BQXP$, then region of $ABCD$ not covered by $APWS$, $BQXP$, and $CRYQ$ is covered by $DSZR$, but that region is not a rectangle. Therefore, it is not possible for $ABCD$ to be divided into 4 rectangles in the prescribed way.

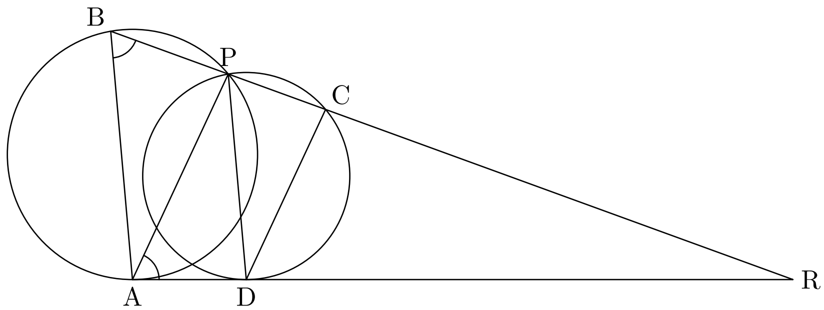


It is possible to divide a rectangle into $n = 5$ rectangles as seen below.



4. A point P is chosen on the side BC of a quadrilateral $ABCD$. The circumcircles of the triangles $\triangle PAB$ and $\triangle PCD$ are both tangent to AD, and $\angle BAP = \angle PDC = 30°$. Find $\angle APD$.

SOLUTION.



Suppose lines AD and BC intersect at point R where D is between A and R. Observe that $\angle ABR = \angle PAR$ as an inscribed angle and angle between tangent and chord with the intercepted arc $\overset{\frown}{AP}$. Let $\angle ABR = \alpha$, then $\angle BAR = 30^\circ + \alpha$ so

$$\angle ARB = 180^\circ - \angle ABR - \angle BAR = 150^\circ - 2\alpha.$$

But we can do the same in the triangle DPR: let $\angle DPR = \beta$. Then $\angle PDR = 30^\circ + \beta$, so $\angle DRP = 150^\circ - 2\beta$.

Now we have $150^\circ - 2\alpha = 150^\circ - 2\beta$. As a result $\alpha = \beta$ and

$$\angle APD = 180^\circ - \angle DPR - \angle ARP - \angle PAR = 180^\circ - \beta - (150^\circ - 2\beta) - \alpha = 30^\circ.$$

5. Find the integer K such that

$$K \leq \frac{1}{\sqrt{2^{2020}}} + \frac{1}{\sqrt{2^{2020}+1}} + \frac{1}{\sqrt{2^{2020}+2}} + \cdots + \frac{1}{\sqrt{2^{2024}-1}} < K+1.$$

SOLUTION.

First note that for any positive number x,

$$\sqrt{x+1} - \sqrt{x} = \left(\sqrt{x+1} - \sqrt{x}\right) \cdot \frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}+\sqrt{x}} = \frac{(x+1)-x}{\sqrt{x+1}+\sqrt{x}} = \frac{1}{\sqrt{x+1}+\sqrt{x}}.$$

From this it is clear that $\sqrt{x+1} - \sqrt{x} > \frac{1}{2\sqrt{x+1}}$ and $\sqrt{x+1} - \sqrt{x} < \frac{1}{2\sqrt{x}}$. In particular, for any $x \geq 1$,

$$2(\sqrt{x+1} - \sqrt{x}) < \frac{1}{\sqrt{x}} < 2(\sqrt{x} - \sqrt{x-1}).$$

Now let A and B be integers with $1 < A < B$. Then

$$\frac{1}{\sqrt{A^2}} + \frac{1}{\sqrt{A^2+1}} + \frac{1}{\sqrt{A^2+2}} + \cdots + \frac{1}{\sqrt{B^2-1}} >$$

$$2\left(\sqrt{A^2+1} - \sqrt{A^2}\right) + 2\left(\sqrt{A^2+2} - \sqrt{A^2+1}\right) + \cdots + 2\left(\sqrt{B^2} - \sqrt{B^2-1}\right) = 2(B-A),$$

and, similarly,

$$\frac{1}{\sqrt{A^2}} + \frac{1}{\sqrt{A^2+1}} + \frac{1}{\sqrt{A^2+2}} + \cdots + \frac{1}{\sqrt{B^2-1}} <$$

$$2\left(\sqrt{A^2} - \sqrt{A^2-1}\right) + 2\left(\sqrt{A^2+1} - \sqrt{A^2}\right) + \cdots + 2\left(\sqrt{B^2-1} - \sqrt{B^2-2}\right) =$$

$$2\left(\sqrt{B^2-1} - \sqrt{A^2-1}\right) < 2\left(\sqrt{B^2} - \sqrt{A^2-A+.25}\right) = 2\left(B - \sqrt{(A+.5)^2}\right) = 2(B-A)+1.$$

Letting $A = 2^{1010}$ and $B = 2^{1012}$ shows that the required value of K is

$$2\left(2^{1012} - 2^{1010}\right) = 2^{1013} - 2^{1011} = 4 \cdot 2^{1011} - 2^{1011} = 3 \cdot 2^{1011}.$$