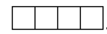
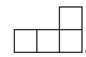
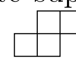
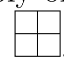
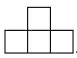
WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET III (2020-2021)

1. We have a 100×100 board built from 10 000 unit squares. We have an infinite supply of tiles of the following form, each made up of four unit squares: , , , , . We fully covered our board without overlap using these tiles (we may rotate or flip them), and for every row and column we counted how many tiles were needed to cover the corresponding row or column. We added up these 200 numbers, and found that the answer is 12,000. How many 2×2 square shape tiles did we use?

SOLUTION. For each tile on the board consider the total number of rows and columns that the tile partially covers. If we add up all these numbers, then we get the sum of the 200 numbers considered in the problem: if a tile covers part of a column (or row), then the covered part contributes 1 to the number of tiles covering the column (or row), and also contributes 1 to the number of columns/rows that the given tile partially covers.

Each square tile covers parts of two rows and two columns, so the number corresponding to this tile is 4. Each other type of tile covers parts of 5 rows and columns. Hence, if there are A squares and B of the other type of tiles, then the total number coming from the tiles is $4A + 5B = 5(A + B) - A$, which is equal to 12,000. There are $\frac{1}{4} \cdot 100 \cdot 100 = 2500$ tiles covering the board (each one is built from 4 squares), hence $A + B = 2500$. This leads to

$$12{,}000 = 5 \cdot 2500 - A, \qquad A = 12{,}500 - 12{,}000 = 500,$$

which means that we used 500 square shaped tiles.

2. Suppose $f(x)$ and $g(x)$ are quadratic polynomials, i.e., $f(x) = ax^2 + bx + c$ and $g(x) = px^2 + qx + r$, where a, b, c, p, q, and r are numbers. We know that the equation $f(x) = g(|x|)$ has four distinct real-valued solutions. How many distinct real-valued solutions does the equation $f(|x|) = g(x)$ have?

SOLUTION. By the assumption, we know that the total number of distinct roots among the two equations $f(x) = g(x)$ (for $x \geq 0$) and $f(x) = g(-x)$ (for $x < 0$) is 4. Note that $f(x) - g(x)$ and $f(x) - g(-x)$ are both polynomials of degree at most 2, so each has at most two roots. Thus, we conclude that $f(x) = g(x)$ has exactly two nonnegative solutions and $f(x) = g(-x)$ has exactly two negative solutions.

Again breaking into cases, the question is equivalent to finding the total number of distinct roots among the two equations $f(x) = g(x)$ (for $x \geq 0$) and $f(-x) = g(x)$ (for $x < 0$). Substituting $y = -x$ into the second equation, we need to find the number of solutions to $f(y) = g(-y)$ for $y > 0$. Since we already know that the polynomial $f(x) - g(-x)$ has two negative roots, it cannot have any positive roots. Thus, we conclude that $f(|x|) = g(x)$ has exactly two distinct solutions (which are both nonnegative).

3. Call a group of 31 integers *nice* if any 12 of the 31 integers can be divided into two groups of 6 so that the sum is the same in the two groups. Show that if 31 integers form a nice group, then they are all equal to each other.

SOLUTION. We first show that a nice group of 31 integers cannot contain both even and odd integers. By assumption the sum of any 12 integers out of the 31 must be even (since it is twice the sum in one of the groups of 6). Suppose that a nice group has both even and odd integers. Choose twelve of them so that we have both an even and an odd integer in this group. Since the sum is even, we must have an even number of odds (at least 2). The remaining $31 - 12 = 19$ integers must have either an odd or an even integer among them, so either switching an odd number from the group to an even number from the 19 (or an even for an odd) will produce 12 integers with an odd sum. This is a contradiction, so the integers in the group must be all even or all odd.

Note that if we have a nice group of 31 integers, then if we subtract the same integer from each number in the group, the resulting 31 integers will also be a nice group. In particular, if m is any number in our group, we can subtract m from each number in our group, and this will result in our having a nice group of integers where 0 is one of the elements in the group. Since the new group contains the number 0, the group must only contain even integers. Note that if we have a nice group of 31 even integers, then if we divide each number in the group by 2, the resulting 31 integers will also be a nice group. But if our nice group contains the number 0, then the resulting nice group will also contain the number 0, so all of the numbers in this nice group will again be even. Therefore, all of the elements in the resulting nice group can be divided by 2 arbitrarily often, from which it follows that all the elements in our resulting nice group must be 0. Hence, our original nice group from which we subtracted the number m must have consisted of 31 numbers all equal to the same m.

4. Suppose for some integer $k > 1$ that $d_1 < \cdots < d_k$ are positive integers, and M is a positive integer that is divisible by all of them. Prove that

$$d_1 d_2 + d_2 d_3 + d_3 d_4 + \cdots + d_{k-1} d_k + d_k d_1 \le M^2.$$

SOLUTION. Each of d_i's is a divisor of M, therefore, $\dfrac{M}{d_i}$ is a positive integer for each i. Also, the inequality on d_i implies that

$$\frac{M}{d_1} > \frac{M}{d_2} > \cdots > \frac{M}{d_k}.$$

Therefore, $\dfrac{M}{d_k} \ge 1$, $\dfrac{M}{d_{k-1}} \ge 2$, and in general, $\dfrac{M}{d_i} \ge k + 1 - i$. Accordingly,

$$d_i \le \frac{M}{k+1-i} \qquad \text{for all } i,$$

and we have

$$d_1 d_2 + d_2 d_3 + \cdots + d_{k-1} d_k + d_k d_1 \le \frac{M}{k} \cdot \frac{M}{k-1} + \frac{M}{k-1} \cdot \frac{M}{k-2} + \cdots + \frac{M}{2} \cdot \frac{M}{1} + \frac{M}{1} \cdot \frac{M}{k}.$$

We claim that the right-hand side is equal to M^2.

Indeed,

$$\frac{M}{k} \cdot \frac{M}{k-1} + \frac{M}{k-1} \cdot \frac{M}{k-2} + \cdots + \frac{M}{2} \cdot \frac{M}{1} + \frac{M}{1} \cdot \frac{M}{k} = M^2 \left(\frac{1}{k(k-1)} + \frac{1}{(k-1)(k-2)} + \cdots + \frac{1}{2 \cdot 1} + \frac{1}{1 \cdot k} \right).$$

Noticing that

$$\frac{1}{a(a-1)} = \frac{1}{a-1} - \frac{1}{a},$$

we rewrite the right-hand side as

$$M^2\left(\frac{1}{k-1}-\frac{1}{k}+\frac{1}{k-2}-\frac{1}{k-1}+\cdots+\frac{1}{1}-\frac{1}{2}+\frac{1}{k}\right).$$

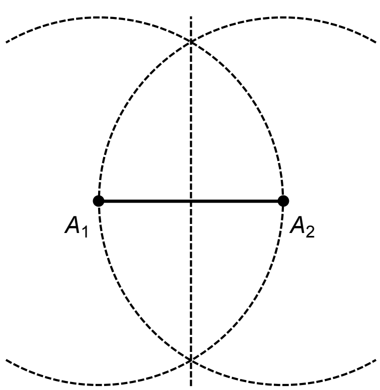
We now see that all of the terms in the parentheses cancel, except for $\frac{1}{1}$, as claimed.

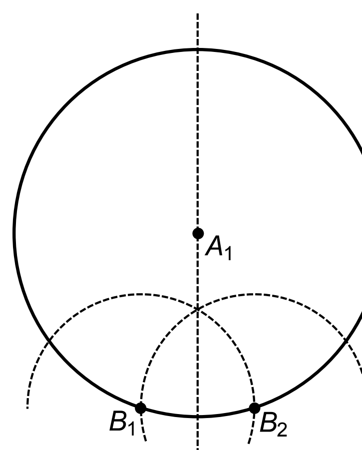
5. Show that if we have 13 points in the plane, we can always choose three of the 13 points that are not the vertices of an isosceles triangle.

SOLUTION. We give a proof by contradiction: suppose that there are 13 points so that any three are the vertices of an isosceles triangle. As a consequence, no three of these points can lie on a straight line. Let A_1 and A_2 be two of the 13 points, and denote the length of $\overline{A_1A_2}$ by r. If $\triangle A_1A_2P$ is an isosceles triangle, then we have either

1. $A_1P = A_2P$, which implies that P is on the perpendicular bisector of $\overline{A_1A_2}$,
2. $A_1A_2 = A_1P$, which means that P is on the circle with center A_1 and radius $r = A_1A_2$, or
3. $A_1A_2 = A_2P$, which means that P is on the circle with center A_2 and radius $r = A_1A_2$.

By assumption there are at least 11 points in the plane which form an isosceles triangle with A_1, A_2, and at most two of them can be on the perpendicular bisector of $\overline{A_1A_2}$. Hence, there are at least 9 points on the circles described in points #2 and #3 above, which means that one of those circles contains at least 5 of the 13 points besides A_1 and A_2. We may assume that this is the circle described in point #2. This means that (possibly considering A_2 as well) we can choose six of the 13 points so they are all on the circle with center A_1 and radius r. We denote this circle by C.





Denote these 6 points on the circle C in counter-clockwise order by $B_1, B_2, \ldots, B_6$, and consider the angles $\angle B_1A_1B_2$, $\angle B_2A_1B_3, \ldots, \angle B_6A_1B_1$. These add up to 360°, and they cannot all be equal to 60° (in this case B_1, A_1, B_4 would be on the same line), so the smallest one is strictly smaller than 60°. Assume that this is the angle $\angle B_1A_1B_2$. Since the triangles $\triangle B_1B_2B_j$, for $j = 3, 4, 5, 6$ are all isosceles, by the same argument as above the points B_3, B_4, B_5, and B_6 are either on the perpendicular bisector of $\overline{B_1B_2}$, the circle with center B_1 and radius B_1B_2, or the circle with center B_2 and radius B_1B_2. The bisector of $\overline{B_1B_2}$ intersects the circle in two points, but only one of them can be among B_3, B_4, B_5, B_6 (since the intersection point on the smaller arc B_1B_2 would produce a central angle which is smaller than $\angle B_1A_1B_2$). The circle with center B_1 and radius B_1B_2 intersects the circle C in two points, and one of them is B_2, so there could be at most one of B_3, B_4, B_5, B_6 on it. The same holds for the circle with center B_2 and radius B_1B_2, which means that there cannot be 4 distinct points B_3, B_4, B_5, B_6 on the circle C so that the triangles $\triangle B_1B_2B_j$, for $j = 3, 4, 5, 6$ are all isosceles. This contradiction shows the statement of the problem is true.