WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET III (2015-2016)

1. We have a list of n different numbers with $n > 1$. We write down the sum of every pair of distinct numbers in our list. Show that we must have at least $2n-3$ different sums.

SOLUTION. Suppose that the n numbers in order are $a_1 < a_2 < \cdots < a_n$. Now consider all sums involving either a_1 or a_n (or both). There are exactly $2n-3$ such sums since there are $n-1$ sums of the form $a_1 + a_k$, $n-1$ sums of the form $a_k + a_{n-1}$, and we counted $a_1 + a_n$ twice. If we show that these $2n-3$ sums are always different, then this will solve the problem.

The sums of the form $a_1 + a_k$ are all different (since the numbers a_k are different). The same is true for the sums of the form $a_k + a_n$, The only case left to check is whether a sum $a_1 + a_k$ (with $k \neq n$) can be the same as a sum $a_n + a_\ell$ (with $\ell \neq 1$). But $a_1 + a_k < a_1 + a_n < a_n + a_\ell$ which shows that $a_1 + a_k$ and $a_n + a_\ell$ cannot be the same. This shows that all $2n-3$ sums are different.

Remarks. 1. It is actually easy to order these $2n-3$ sums:

$$a_1 + a_2 < a_1 + a_3 < \cdots < a_1 + a_n < a_2 + a_n < a_3 + a_n < \cdots < a_{n-1} + a_n.$$

2. It is not hard to check that the statement is not true with a number bigger than $2n-3$: if our numbers are 1, 2, $\ldots n$ then the sums give the integers between 3 and $2n-1$, and the number of these is exactly $2n-3$.

2. We have n numbers $x_1, \ldots, x_n$ such that $0 < x_i < 1$ for each i, and $n > 1$. Show that

$$2 < (1-x_1)(1-x_2)\cdots(1-x_n) + (1+x_1)(1+x_2)\cdots(1+x_n) < 2^n.$$

SOLUTION. Imagine that we expand the second product. When we do this, we have to multiply out all the terms, and grouping the terms we get

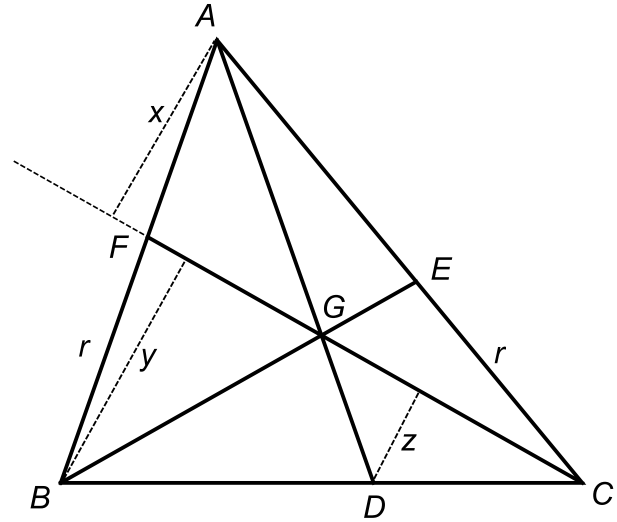
$$\begin{aligned}(1+x_1)(1+x_2)\cdots(1+x_n) = &1 + \underbrace{x_1 + x_2 + \cdots + x_n}_{\text{single terms}} + \\ &\underbrace{x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n}_{\text{products with two terms}} + \cdots + \underbrace{\qquad\qquad\qquad}_{\text{products with } k \text{ terms}} + \\ &\cdots + \underbrace{x_1x_2\cdots x_n}_{\text{product with } n \text{ terms}} .\end{aligned}$$

When we do the same with the first product, we get the same expression, except any product involving an *odd* number of x_k will get an extra negative sign. When we add the two expressions the products with an odd number of factors will cancel each other out and we get

$$2 + 2\underbrace{(x_1x_2 + x_1x_3 + \cdots \qquad\qquad\qquad\qquad)}_{\text{products with even number of terms}}.$$

Since each x_k is positive, this expression is bigger than 2. On the other hand, each x_k is less than 1, so if we replace each of them with this value, then the expression will become bigger. (Here we used the fact that each product of x_k will have a + sign in front of it.) Substituting $x_k = 1$ for all k gives exactly 2^n in the original expression, which shows the second part of the inequality.

3. In $\triangle ABC$, $AB = 9$, $BC = 10$, and $AC = 11$. Points D, E, and F lie on sides $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively, so that segments $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ all intersect at G as shown. Given that $BF = CE$ and $AG : DG = 2 : 1$, find BF.



SOLUTION. Let $r = BF = CE$. Let x, y, and z be the distances to line CF of points A, B, and D, respectively. Then by similar triangles $\frac{AF}{BF} = \frac{x}{y}$, $\frac{CD}{BC} = \frac{z}{y}$, and $\frac{DG}{AG} = \frac{z}{x}$. This shows that $\frac{CD}{BC} = \frac{DG}{AG} \cdot \frac{AF}{BF} = \frac{1}{2} \cdot \frac{9-r}{r}$. Similarly, (by considering the distance of A, C, and D to the line BE) we get that $\frac{BD}{BC} = \frac{1}{2} \cdot \frac{11-r}{r}$.
Adding these two together gives $\frac{CD}{BC} + \frac{BD}{BC} = \frac{BC}{BC} = 1 = \frac{1}{2}\left(\frac{9-r}{r} + \frac{11-r}{r}\right)$, so $2r = 20 - 2r$ and $BF = CE = r = 5$.

Note that the length of side $\overline{BC}$ does not affect the result.

Remark. As a side result, we basically proved van Aubel's Theorem for triangles. This theorem states that if G is inside the triangle $\triangle ABC$ and the lines AG, BG, CG intersect the sides of the triangle at the points D, E and F (just like in the picture), then

$$\frac{AG}{GD} = \frac{AE}{EC} + \frac{AF}{FB}.$$

In our case this gives $2 = \frac{9-r}{r} + \frac{11-r}{r}$ which yields $r = 5$.

4. We define a sequence of integers $a_1, a_2, \ldots, a_n, \ldots$ successively, by setting $a_1 = 2$ and $a_{n+1} = a_n^2 - a_n + 1$ for $n \geq 1$. The first few elements are $a_1 = 2, a_2 = 3, a_3 = 7, a_4 = 43$. Show that in the sequence we cannot find two different integers with a common divisor that is bigger than one.

SOLUTION. We will prove by induction that $a_n = a_{n-1} \cdot a_{n-2} \cdots a_1 + 1$ for $n \geq 2$. We have $a_2 = a_1 + 1$. If $a_n = a_{n-1} \cdot a_{n-2} \cdots a_1 + 1$, then $a_{n+1} = a_n^2 - a_n + 1 = a_n(a_n - 1) + 1 = a_n(a_{n-1} \cdot a_{n-2} \cdots a_1) + 1$, which completes the induction. Suppose, for the sake of contradiction, that an integer d divides both a_n and a_{n+k} with $d > 1$ and $k \geq 1$. Then the integer d must divide the difference $a_{n+k} - a_{n+k-1} \cdots a_n \cdots a_1 = 1$, which means that d divides 1, a contradiction. So we cannot find two different integers in the sequences with a common divisor that is bigger than one.

5. Anne and Bert play the following game. They start with a $1 \times n$ grid of n unit squares which are initially white. Anne starts the game, and they take steps one after another. In each step the appropriate player can change either one or two neighboring white squares to black. The player who colors the last available square(s) wins. For which n will Anne have a winning strategy?

SOLUTION. We will show that Anne always has a winning strategy. It is clear that this is the case for $n = 1$ or $n = 2$, since in these cases Anne can change all the squares to black in the first step.

Suppose that $n > 2$. In the case that n is odd, then $n = 2k + 1$ for some integer k. In that case Anne can color the $(k+1)^{th}$ (the middle) square black, and we are left with two blocks of $1 \times k$ white squares left. Imagine that we have a mirror between the two blocks exactly in the middle (so that the two blocks are mirror images of each other), and suppose that Anne follows the following strategy: if Bert colors one or two squares from one of the blocks, then Anne does the same with the mirror image(s) of these square. This way after Anne's each move, the remaining white squares in the two block will be mirror images of each other (if anything is left), which means that Bert can never take the last square(s).

In the case that n is even, then $n = 2k$ for some integer k. Then Anne will win if she colors the two middle squares (k^{th} and $(k+1)^{th}$) black. This way we have two blocks of $1 \times (k-1)$, and Anne can again follow the 'mirror' strategy described as before.