WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET II (2020-2021)

1. Find all positive integer pairs (m, n) satisfying $m^2 - n^2 - 2n = 2021$.

SOLUTION. Subtracting 1 from each side of the given equation yields

$$2020 = m^2 - (n^2 + 2n + 1) = m^2 - (n+1)^2 = (m - n - 1)(m + n + 1).$$

Note that the factors $m - n - 1$ and $m + n + 1$ differ by $2n + 2$, which is even, implying that both $m - n - 1$ and $m + n + 1$ must be even in order for their product to be the even number 2020. Since $m + n + 1$ is positive, this must be true for $m - n - 1$ as well, and we have $0 < m - n - 1 < m + n + 1$.

The prime factorization of 2020 is $2 \cdot 2 \cdot 5 \cdot 101$. If 2020 is the product of two even numbers $2a$ and $2b$ with $0 < a < b$ then $a \cdot b = 5 \cdot 101$, and either $a = 1, b = 5 \cdot 101$ or $a = 5, b = 101$. This gives the following two possibilities for $m - n - 1$ and $m + n + 1$:

$$\begin{aligned} m - n - 1 &= 2 \\ m + n + 1 &= 1010 \end{aligned} \qquad\qquad \begin{aligned} m - n - 1 &= 10 \\ m + n + 1 &= 202. \end{aligned}$$

The corresponding solutions for (m, n) are $(506, 503)$ and $(106, 95)$.

2. Show that for every nonnegative number x, we have $x + 3 \geq 4\sqrt[4]{x}$.

SOLUTION. Since x is nonnegative, there is a well-defined nonnegative number $y = \sqrt{x}$. Then the inequality becomes $y^2 + 3 \geq 4\sqrt{y}$. Since both sides are nonnegative, it is sufficient to prove the square of the inequality:

$$y^4 + 6y^2 + 9 - 16y \geq 0.$$

Notice that the left-hand side is always nonnegative because it can be rewritten as

$$(y^2 - 1)^2 + 8(y - 1)^2,$$

which finishes the proof.

Alternate solution. Let $a = \sqrt[4]{x}$. Then we need to show that for $a \geq 0$, we have

$$a^4 + 3 \geq 4a.$$

We will show that this inequality actually holds for all a. Note that for $a = 1$, the two sides are equal, which suggests that we should introduce the variable $b = a - 1$. Then $a = b + 1$, and we need to show that

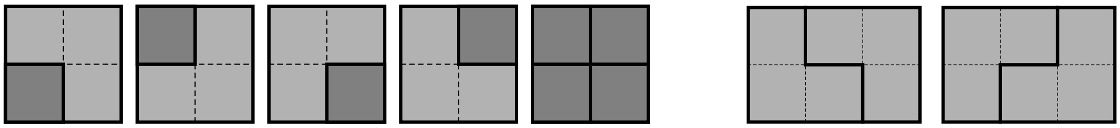
$$(b + 1)^4 + 3 - 4(b + 1) \geq 0.$$

Expanding the fourth power gives

$$(b + 1)^4 + 3 - 4(b + 1) = b^4 + 4b^3 + 6b^2 + 4b + 1 + 3 - 4(b + 1) = b^4 + 4b^3 + 6b^2,$$

and $b^4 + 4b^3 + 6b^2 = b^2(b^2 + 4b + 6)$. But $b^2 + 4b + 6 = (b + 2)^2 + 2 \geq 2$. Hence, $b^2(b^2 + 4b + 6) \geq 0$.

3. We have a $2 \times n$ board made up of $2n$ unit squares. We also have an unlimited supply of two types of tiles: 1×1 tiles, and L-shaped tiles composed of three 1×1 tiles. Let a_n denote the total number of ways to fully cover the $2 \times n$ board with the available tiles in a single layer. (E.g., $a_2 = 5$, as demonstrated in the figure below.) Prove that $a_n \le 3^n$.



SOLUTION. We have $a_1 = 1$ (since we can only use the 1×1 tiles in this case), and $a_2 = 5$ (as the figure shows). We set $a_0 = 1$. We will show that for $n \ge 3$, we have

$$a_n = a_{n-1} + 4a_{n-2} + 2a_{n-3}. \qquad (*)$$

Imagine that we number the columns of the $2 \times n$ grid from left to right, and consider the first three columns.

If the first column is covered by two 1×1 tiles, then deleting the the first column, we get a covering of the $2 \times (n-1)$ board. Hence, we have a_{n-1} such tilings.

If the first column is not covered fully with two 1×1 tiles, then part of the first column is covered by an L-shaped tile. This L-shaped tile covers three squares from the first two columns. If the fourth square of the first two columns is covered by a 1×1 tile, then removing the first two columns, we get a covering of the $2 \times (n-2)$ board. The first two columns can be covered 4 different ways with an L-shaped tile and a unit tile (see the first four tilings in the figure above). Hence, this gives $4a_{n-2}$ tilings that we haven't counted before.

The only remaining tilings that we haven't counted are the ones where there is an L-shaped tile covering three squares of the first two columns, and the fourth square is covered by another L-shaped tile. This fourth square then must be in the second column, hence, the L-shaped tile covering this square fully covers the third column. We have two possible configurations for this (see the two tilings on the right in the figure above), and removing the first three columns, we get a covering of the $2 \times (n-3)$ board. Hence, the number of these tilings is $2a_{n-3}$. (In case of $n = 3$, there is no board left after removing the first three columns. But since we set $a_0 = 1$, our statement still holds, $2a_0 = 2$ gives the number of tilings considered in this case.)

This proves the identity $(*)$. To prove that $a_n \le 3^n$, we use induction. The inequality holds for $n = 0$, $n = 1$, and $n = 2$. Assume that $a_k \le 3^k$ for every $0 \le k < n$, for some $n \ge 3$. Then by $(*)$, we have

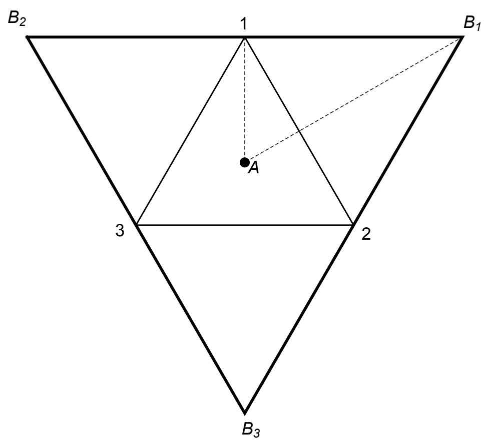
$$a_n = a_{n-1} + 4a_{n-2} + 2a_{n-3} \le 3^{n-1} + 4 \cdot 3^{n-2} + 3 \cdot 3^{n-3} = (9 + 12 + 1) \cdot 3^{n-3} = 22 \cdot 3^{n-3} < 27 \cdot 3^{n-3} = 3^n.$$

This proves the induction step and finishes the proof.

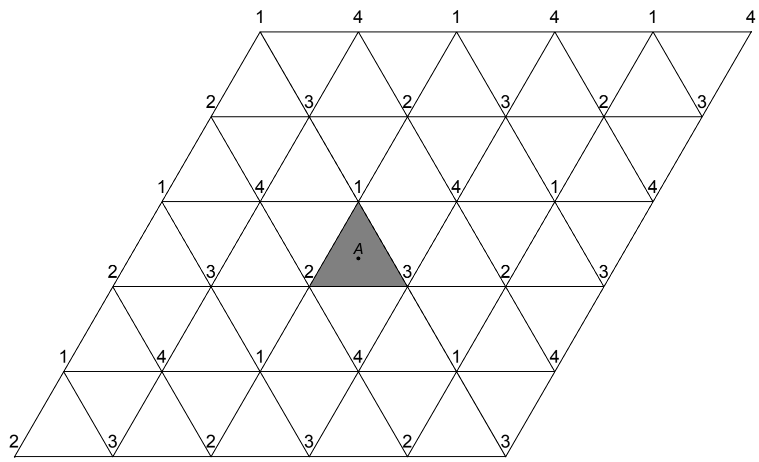
4. A strange new planet has been been discovered, shaped like a regular tetrahedron with edge length 900 miles. (A regular tetrahedron has four faces which are all equilateral triangles.) The entire surface of the planet is covered by oceans. Astronomers observe that a massive earthquake has triggered a tsunami whose waves are traveling along the surface at the speed of 300 miles per hour. If the epicenter of the earthquake is at the centroid of a face, how long does it take for the waves to reach the vertex opposite to this face?

SOLUTION. Label the vertices of the tetrahedron 1, 2, 3, and 4. Without loss of generality, let the epicenter of the tsunami be the point A located at the centroid of $\triangle 123$. Let d be the length (in miles) of the shortest path from A to vertex 4 in miles, where the path travels on the surface of the tetrahedron; the tsunami will reach point 4 in $d/300$ hours.

The shortest path between two points in a plane is a straight line. But how can we compute the shortest path length between two points on the surface of the tetrahedron? A natural approach is to "flatten" the surface by folding down the faces $\triangle 124$, $\triangle 234$, and $\triangle 134$ in the plane of $\triangle 123$. In this way, we obtain an equilateral triangle with edges twice as long as those of $\triangle 123$, and the vertex 4 is "copied" to the vertices of the new equilateral triangle, which we denote by B_1, B_2, and B_3 (see Figure (a)). Within $\triangle B_1B_2B_3$, the shortest path between A and one of the vertices $\{B_1, B_2, B_3\}$ is the line segment connecting A to that point. This line segment is the hypothenuse of a 30-60-90 triangle with longer leg of length 900, so it has length $900 \cdot \frac{2}{\sqrt{3}} = 600\sqrt{3}$. Thus, it seems that the shortest path length between A and vertex 4 should be $d = 600\sqrt{3}$, and the tsunami will hit vertex 4 after $2\sqrt{3} \approx 3.46$ hours (roughly 3 hours and 38 minutes).



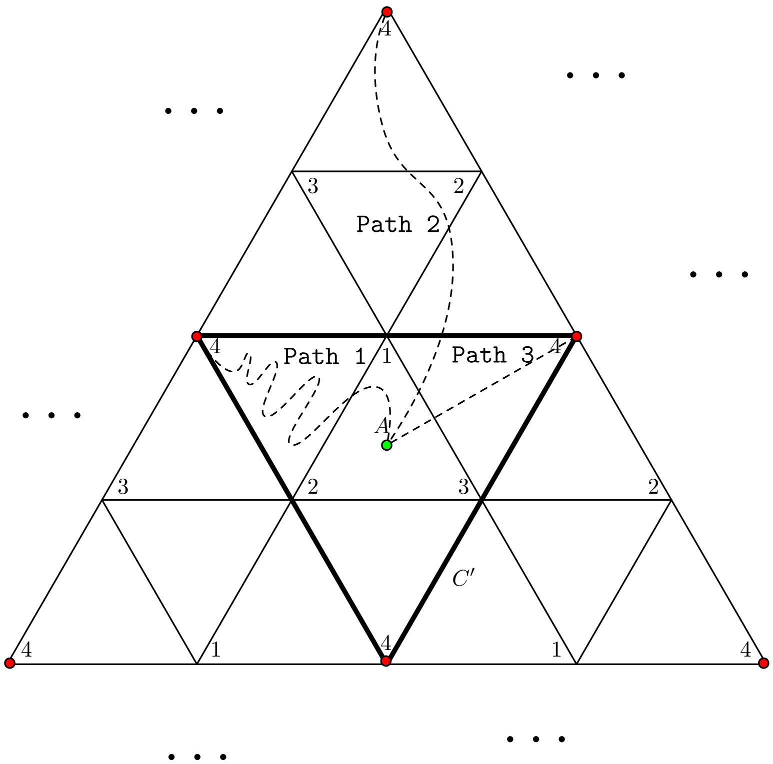
(a)



(b)

However, our solution is not entirely complete: We have identified the shortest path from A to vertex 4 which passes through two faces ($\triangle 123$ and one of the other three faces), but we also need to show that it is impossible to find a path winding through more than two faces which is even shorter. To this end, we note that any path on the surface of the tetrahedron can be "unwrapped" to a path in the plane. Imagine that we place the tetrahedron on a plane $\mathcal{P}$, so that $\triangle 123$ is at the bottom. Consider a path connecting A to 4, and imagine that we are traveling from A toward 4. We will start out on the face $\triangle 123$ (which is in the plane $\mathcal{P}$), but at some point, we will move to a neighboring face. When we do that, we "flip" the tetrahedron over the edge, so that the current face also rests on $\mathcal{P}$. This way, the part of the path in the current face is again in the plane $\mathcal{P}$. We can continue this: each time we move to a new face of the tetrahedron, we flip the tetrahedron so the current part rests on the plane $\mathcal{P}$. The path on the surface of the tetrahedron has thus been copied ("unwrapped") to a path on the plane $\mathcal{P}$. Note that each time we "flip" the tetrahedron, the position of the face resting on $\mathcal{P}$ will be an equilateral triangle from the triangular tiling of the plane, and we can also follow the position of the labeled vertices (see Figure (b)). The rule is the following: any two equilateral triangle tiles that share a side must have all four labels among their vertices.

Any path connecting A to vertex 4 will be converted to a path in the plane $\mathcal{P}$ connecting A to one of the lattice points labeled by 4 in the tiling. Moreover, any path from A to *any* of the points labeled 4 in the triangular lattice corresponds to a path (of the same length) from A to vertex 4 in the original tetrahedron.



(c)

Figure (c) shows three possible paths from point A to three different points labeled 4; all can be "folded back" to a unique path on the surface of the tetrahedron.

The shortest path from point A to some fixed point labeled 4 on the tiling in Figure (b) is the straight line joining the two points. Thus, the shortest distance between point A and vertex 4 on the surface of the tetrahedron is given by calculating the length of the shortest line segment joining point A and any point labeled 4 in the tiling. These are exactly the points that are on the triangles that are neighbors to the original $\triangle 123$, which means that our first guess indeed found the shortest path from A to 4 on the surface of the tetrahedron.

Note: Our solution used the fact that we have a regular tetrahedron. This way, the "unwrapped" faces of the tetrahedron form a non-overlapping tiling, as described above. This would not be the case if the four faces of the tetrahedron were not equilateral triangles.

5. We have 2020 distinct numbers arranged in a circle. We say that a pair of numbers A, B is *dominating* if A and B are not next to each other on the circle, and on one of the two arcs between A and B, all numbers are smaller than both A and B. (If all numbers on both arcs happen to be smaller than A and B, we still call the pair dominating.) Find the number of dominating pairs, and show that the answer is the same, no matter how the numbers are arranged.

SOLUTION. There are 2017 dominating pairs. More generally, suppose n numbers are arranged in a circle (instead of 2020), where $n > 2$. Let us prove that there are exactly $n - 3$ dominating pairs.

We proceed by induction. In the base case, $n = 3$ and there is nothing to prove: all pairs are adjacent, and therefore none are dominating.

For the induction step, suppose the claim holds for $n = k$, and let us prove it for $n = k + 1$. Let A be the smallest number in the circle. Clearly, it cannot be part of a dominating pair (because it is smaller than all the others). Moreover, if B and C are its neighbors, they form a dominating pair (because one of the two arcs between B and C contains only one number, A, which is smaller than both of them).

Let us now remove A from the circle, so that there are now k numbers left. This will not affect which pairs are dominating, with a single exception: B and C become adjacent, so they no longer form a dominating pair. By the induction hypothesis, there are $k - 3$ dominating pairs left, and therefore, there were $k - 3 + 1 = n - 3$ before we removed A.

Remark. There is a picture behind this problem: if we place the numbers at the vertices of a regular n-gon, and draw the diagonal connecting each dominating pair, the result partitions the polygon into $n - 2$ triangles because none of these diagonals will cross.