WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET I (2020-2021)

1. You have 2020 \$1 bills and 11 envelopes. Show that it is possible to divide the bills among the envelopes in such a way that you could give someone any exact integer amount from \$1 to \$2020 by handing over some of the envelopes (at least one, at most 11). Then show that this would be impossible if you only had 10 envelopes.

SOLUTION. Divide the money into 11 envelopes, as follows:

$$1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 997.$$

We first note that using the first 10 envelopes, it is possible to make any sum of money from 1 to 1023. Indeed, the binary representation of any integer between 1 and 1023 corresponds to a selection of envelopes, e.g., $1 = 0000000001$ (take just the first envelope), $10 = 0000001010$ (take the second and fourth envelopes), or $1023 = 1111111111$ (take all ten envelopes). In order to make $\$x$, where $1024 \leq x \leq 2020$ we can simply take the eleventh envelope and then use the binary representation of $x - 997$ to select the remaining envelopes.

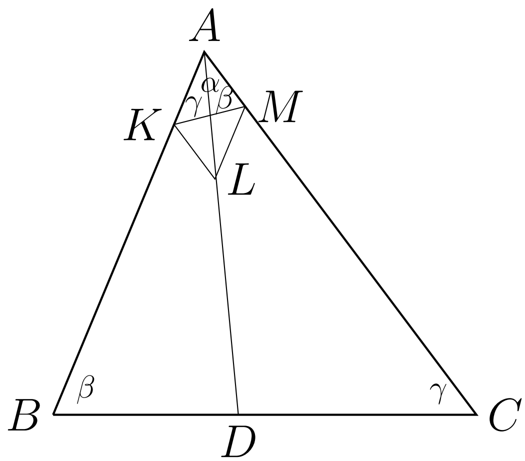
Now assume that our task would be possible with 10 envelops with a certain distribution of the \$1 bills. That would mean that each integer value between 1 and 2020 could be expressed as the sum of the contents of certain envelops in at least one way. But each selection of envelopes corresponds to deciding for each of the 10 envelopes whether that envelope is to be used or not. Thus, there are two choices possible for each envelope (to use it or not to use it), and this means there are at most $2^{10} = 1024$ collections of envelopes one could choose. (In fact, we have only 1023 choices, since we cannot choose 0 envelopes.) Because 1024 is less than the needed 2020, this is a contradiction and proves that the task is impossible with 10 envelops.

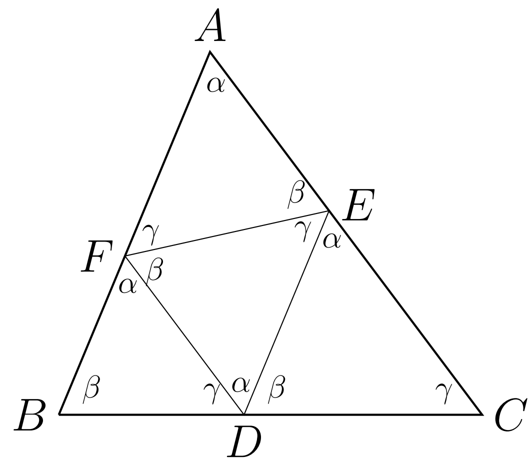
2. In a group of 300 people, each person is friends with exactly 200 others. (Nobody is their own friend, and if A is B's friend, then B is A's friend.) However, no four people are all friends with each other. Show that it is possible to find a group of 100 people in which no two are friends.

SOLUTION. Choose a person A, and let B be any of his friends. Note that B is friends with 200 people, and there are exactly 100 people who are not A's friends (including A himself), therefore, out of B's friends, at least 100 are also friends with A. Therefore, there is a group of 100 people who are friends with both A and B. In this group, no two people are friends, as required. Indeed, if this group contained two friends, C and D, then A, B, C, and D would all be friends with each other, contrary to the problem's statement.

3. Prove that any non-equilateral triangle can be divided into four similar triangles such that the four triangles are not all congruent to each other.

SOLUTION. Let the triangle be $\triangle ABC$ where the angles at vertices A, B, and C, are α, β, and γ, respectively, and assume that $\beta \neq \gamma$. Let K be a point on $\overline{AB}$ and M be a point on $\overline{AC}$ such that $\angle AKM = \gamma$. Thus, $\triangle AMK$ is similar to $\triangle ABC$. Let L be the point such that $\overline{KL}$ is parallel to $\overline{AC}$ and $\overline{ML}$ is parallel to $\overline{AB}$ making $AKLM$ a parallelogram. Let D be the intersection of $\overline{BC}$ and line AL. Let E be the point on $\overline{AC}$ such that $\overline{DE}$ is parallel to $\overline{AB}$, and let F be the point on $\overline{AB}$ such that $\overline{DF}$ is parallel to $\overline{AC}$. Because $\overline{KL}$ and $\overline{DF}$ are both parallel to $\overline{AC}$, and $\overline{ML}$ and $\overline{DE}$ are both parallel to $\overline{AB}$, it follows that $\overline{KL} \parallel \overline{DF}$ and $\overline{ML} \parallel \overline{DE}$. Thus, $AK/AF = AL/AD = AM/AE$, and this shows that $\triangle AKM$ is similar to $\triangle AFE$. It follows that $\angle AEF = \beta$ and $\angle AFE = \gamma$.





Because $\overline{DE}$ is parallel to $\overline{AB}$, $\angle DEC = \alpha$ and $\angle EDC = \beta$. Similarly, because $\overline{DF}$ is parallel to $\overline{AC}$, $\angle DFB = \alpha$ and $\angle FDB = \gamma$. Because the three angles in a triangle add to 180° and three angles whose sum forms a straight angle add to 180°, it follows that $\angle EDF = \alpha$, $\angle DFE = \beta$, and $\angle DEF = \gamma$. Therefore, segments $\overline{DE}$, $\overline{EF}$, and $\overline{DF}$ divide $\triangle ABC$ into four similar triangles. Because $\beta \neq \gamma$, it follows that $AB \neq AC$. But $\overline{DE}$ is a side of $\triangle EDC$ and a side of $\triangle DFE$, so these two triangles must not be congruent to each other. Similarly, $\triangle FBD$ and $\triangle DFE$ cannot be congruent to each other. Thus, this is a decomposition of $\triangle ABC$ into four similar triangles not all of which are congruent. Note that $\triangle AEF$ is congruent to $\triangle DFE$. Also note that the only way to decompose an equilateral triangle into four similar triangles is to make all four triangles congruent.

4. The positive numbers a, b and c have a sum of 1. Show that

$$(ab + bc + ac)^2 \geq 3abc.$$

SOLUTION.

Since $a + b + c = 1$, the inequality is equivalent to

$$(ab + bc + ac)^2 \geq 3abc(a + b + c).$$

Making a change of variables, $ab = x,\ bc = y$, and $ca = z$ the last inequality becomes

$$(x + y + z)^2 \geq 3(xy + yz + zx).$$

Expanding the left hand side and canceling the double products we see that the last inequality is equivalent to

$$x^2 + y^2 + z^2 \geq xy + yz + zx.$$

To prove this inequality, notice that for any real numbers u and v, $(u - v)^2 \geq 0$ and, therefore, $u^2 + v^2 \geq 2uv$. Hence,

$$\frac{1}{2}(x^2 + y^2) \geq xy,\ \frac{1}{2}(y^2 + z^2) \geq yz \text{ and } \frac{1}{2}(z^2 + x^2) \geq zx.$$

Adding these three inequalities together we finish the proof.

Note that we did not use that a, b, and c are positive, so the statement holds for any three numbers that add up to 1.

5. Find all positive integers a and b so that $a^2 - 4b$ and $b^2 - 4a$ are both perfect squares.

SOLUTION. Suppose that a and b are positive integers and $a^2 - 4b$ and $b^2 - 4a$ are both perfect squares.

Let's assume first that $b \leq a$. The numbers $a^2 - 4b$ and a^2 have the same parity since their difference is $4b$, an even number. $a^2 - 4b$ is a perfect square that is smaller than a^2 that cannot be $(a-1)^2$, since a^2 and $(a-1)^2$ have different parities. Hence,

$$a^2 - 4b \leq (a-2)^2 = a^2 - 4a + 4.$$

This implies $4b \geq 4a - 4$ and $b \geq a - 1$. Because we assumed $b \leq a$ this implies $b = a$ or $b = a - 1$.

If $b = a$, then $a^2 - 4a < a^2 - 4a + 4 = (a-2)^2$ has to be a perfect square with the same parity as a^2, which means that $a^2 - 4a \leq (a-4)^2$. This leads to $4a \leq 16$ and $a \leq 4$. $a = b = 4$ provides a solution (since in this case $a^2 - 4a = 0$), but if $1 \leq a \leq 4$, then $a^2 - 4a = (a-2)^2 - 4 < 0$ and, hence, $a^2 - 4a$ cannot be a perfect square.

If $b = a - 1$, then we need $a^2 - 4(a-1)$ and $(a-1)^2 - 4a$ to be perfect squares. We have

$$(a-1)^2 - 4a = a^2 - 6a + 1 < a^2 - 6a + 9 = (a-3)^2$$

and since $(a-1)^2 - 4a$ is a perfect square with the same parity as $(a-1)^2$, we must have

$$a^2 - 6a + 1 = (a-1)^2 - 4a \leq (a-5)^2 = a^2 - 10a + 25.$$

This leads to $4a \leq 24$ and $a \leq 6$. The value $a = 6$ leads to a solution, in this case $b = a - 1 = 5$, $a^2 - 4b = 16$ and $b^2 - 4a = 1$. If $1 \leq a \leq 5$, then $(a-1)^2 - 4a = (a-3)^2 - 8 < 0$, and, hence, these cases do not lead to solutions.

This means that in the $b \leq a$ case the only solutions are $a = 4, b = 4$, and $a = 6, b = 5$. By reversing the roles of a and b we get the additional solution $a = 5, b = 6$.