WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET V (2019-2020)

1. The product of four consecutive integers $a, a+1, a+2$ and $a+3$ can also be written as a product of two consecutive integers. What are the possible values of a?

SOLUTION. Denote by a the smallest of the four consecutive integers, then the four numbers are $a, a+1, a+2, a+3$. Suppose that the product of these are equal to the product of two consecutive integers that we denote by b and $b+1$:

$$a(a+1)(a+2)(a+3) = b(b+1). \tag{1}$$

If $-3 \leq a \leq 0$ then $a(a+1)(a+2)(a+3) = 0$. In these cases we can choose b to be -1 or 0 and the two sides are equal.

If $a < -3$ then all 4 integers are negative, and their product is the same as the positive integers $-a-3, -a-2, -a-1, -a$. Thus if for an $a < -3$ the product of the four integers can be written as $b(b+1)$ then the same is true for the four integers starting with $1 \leq -a-3$, which means we may assume that $a \geq 1$.

If $a \geq 1$, then $a(a+1)(a+2)(a+3) > 0$. Therefore, $b(b+1)$ must be positive as well. We can assume without loss of generality that $b \geq 0$: indeed, if $b < 0$, we get the same product from the numbers $-b-1, -b$, so we can replace b with $-b-1 \geq 0$. Let us show that is impossible for $a \geq 1$ and $b \geq 0$.

Note that

$$a(a+1)(a+2)(a+3) = a(a+3) \cdot (a+1)(a+2) = (a^2+3a)(a^2+3a+2).$$

If $b \leq a^2 + 3a$ then

$$b(b+1) \leq (a^2+3a)(a^2+3a+1) \leq (a^2+3a)(a^2+3a+2).$$

If $b \geq a^2 + 3a + 1$ then

$$b(b+1) \geq (a^2+3a+1)(a^2+3a+2) \geq (a^2+3a)(a^2+3a+1).$$

Hence for a positive integer a we cannot find a positive integer b so that equation () holds.

2. At time 0, a Petri dish contains one red and one blue amoeba. At the end of each minute, exactly one amoeba in the dish divides into two identical amoebas, each with the same color as its parent amoeba. Thus, at the end of n minutes, the dish contains $n+2$ amoebas. Assume that the amoeba that divides into two is always chosen uniformly at random (i.e., it is equally likely that any amoeba is chosen). Find the probability that after 2020 minutes, the Petri dish contains more red than blue amoebas.

SOLUTION. Suppose that after n minutes the Petri disk contains k red amoebas and $n+2-k$ blue amoebas. Then after $n+1$ minutes the dish will contain $k+1$ red amoebas and $n+2-k$ blue amoebas with probability $\frac{k}{n+2}$ and k red amoebas and $n+3-k$ blue amoebas with probability

$\frac{n+2-k}{n+2}$. Thus, the probability that a particular sequence of colors are chosen for the 2020 minutes is

$$\frac{a_1}{2} \cdot \frac{a_2}{3} \cdot \frac{a_3}{4} \cdot \frac{a_4}{5} \cdots \frac{a_{2020}}{2021},$$

where a_n is the number of amoebas in the dish at time $n-1$ of the color chosen at time n. If the sequence of amoebas chosen resulted in there being exactly 1011 red amoebas and 1011 blue amoebas in the dish, then for each k between 1 and 1010, there was a time when there were k red amoebas in the dish and a red amoeba divided, and a time when there were k blue amoebas in the dish and a blue amoeba divided. Thus, the probability that this sequence of color choices happens is

$$\frac{1 \cdot 1 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 4 \cdot 4 \cdot \cdots 1010 \cdot 1010}{2 \cdot 3 \cdot 4 \cdot \cdots \cdot 2021}.$$

Also, if the dish ends up with 1011 red amoebas and 1011 blue amoebas, then there were exactly 1010 times out of the 2020 minutes when a red amoeba was chosen to divide. There are $\binom{2020}{1010}$ ways these 1010 times could be chosen. Thus, the probability that at the end of 2020 minutes there are exactly 1011 red amoebas and 1011 blue amoebas in the dish is

$$\binom{2020}{1010} \cdot \frac{1 \cdot 1 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 4 \cdot 4 \cdot \cdots 1010 \cdot 1010}{2 \cdot 3 \cdot 4 \cdot \cdots \cdot 2021} = \frac{2020!}{1010! \cdot 1010!} \cdot \frac{(1010!)^2}{2021!} = \frac{1}{2021}.$$

It follows that the probability that after 2020 minutes the number of red and blue amoebas differ is $1 - \frac{1}{2021} = \frac{2020}{2021}$. Since the probability that there will be more red amoebas than blue amoebas at that time is equal to the probability that there will be more blue amoebas than red amoebas, so the probability that there will be more red amoebas is half of $\frac{2020}{2021}$ which is $\frac{1010}{2021}$.

Alternate solution. We will show that after k minutes the probability that there are r red and $k+2-r$ blue amoebas is equal to $\frac{1}{k+1}$ for all $1 \le r \le k+1$.

We use induction. The statement is true for $k=0$, since we have 1 red and 1 blue amoebas at the beginning (and r can only be 1). Now assume that the statement is true for some k, we would like to show it for $k+1$. First assume that $2 \le r \le k+1$ then both r and $r-1$ are between 1 and $k+1$. We can get r red amoebas after $k+1$ minutes by

- having r red amoebas after k minutes, and choosing a blue amoeba in the last step
- having $r-1$ red amoebas after k minutes, and choosing a red amoeba in the last step.

By the induction hypothesis $\frac{1}{k+1}$ of the possible sequence of moves in the first k minutes results in r red amoebas, and then out of the $k+2-r$ of the $k+2$ choices would give a blue amoeba in the last step: this is $\frac{1}{k+1} \cdot \frac{k+2-r}{k+2}$ of all possible sequence of moves in the first $k+1$ minutes. Similarly, $\frac{1}{k+1}$ of the possible sequence of moves in the first k minutes results in $r-1$ red amoebas, and then $r-1$ out of the $k+2$ choices would give a red amoeba in the last step. This is $\frac{1}{k+1} \cdot \frac{r-1}{k+2}$ fraction of all possible moves. Hence the proportion of the outcomes which results in r red amoebas after $k+1$ steps is

$$\frac{1}{k+1} \cdot \frac{k+2-r}{k+2} + \frac{1}{k+1} \cdot \frac{r-1}{k+2} = \frac{k+1}{(k+1)(k+2)} = \frac{1}{k+2},$$

which means that the corresponding probability is $\frac{1}{k+2}$.

If $r=1$ then we must have 1 red after k minutes and we chose a blue in the last step, and when $r=k+1$ then we must have k reds after k minutes and we chose a red in the last step. These cases can be handled similarly, and we get that the corresponding probabilities are

$$\frac{1}{k+1}\frac{k+1}{k+2} = \frac{1}{k+1}.$$

This completes the induction step and proves the statement. The probability of having more red than blue amoebas after 2020 minutes is the sum of probabilities of having 1011, 1012, ..., or 2021 amoebas, which is $1011 \cdot \frac{1}{2021} = \frac{1011}{2021}$.

3. Suppose that a point P inside a given convex quadrilateral $ABCD$ has the property that the areas of the triangles PAB, PBC, PCD, and PAD are all equal. Show that one of the diagonals of $ABCD$ must divide the quadrilateral into two triangles with equal areas.

SOLUTION. Let us show that P lies on one of the two diagonals of the quadrilateral (AC or BD).

The triangles PAB and PAD share the side PA. Since their areas are equal, the heights dropped from B and D onto PA must be equal. In other words, B and D are equidistant from PA (but on the opposite sides). This implies that the line AP must pass through the midpoint of BD. Similarly, looking at the triangles PCB and PCD, we see that the line PC must pass through the midpoint of BD as well. This leaves two possibilities: either P is the midpoint of BD (and in particular P is on the diagonal BD) or the two lines PA and PC pass through two distinct points (P and the midpoint of BD), which means that the two lines PA and PC coincide. In the latter case, the three points P, A, and C are collinear, and therefore P belongs to the diagonal of AC. In either case, P belongs to one of the two diagonals, as claimed.

The statement now follows. For example, if P lies on the diagonal AC, then each of the triangles ABC and ACD is split into two triangles of equal area (PAB and PBC, and PCD and PDA), and therefore ABC and ACD have equal area as well. (In fact, it is easy to see that P must be the midpoint of one of the diagonals.)

4. In a class of n students there are seven clubs. A student can be a member of multiple clubs, and each club has at least one members. Somebody noticed that if we choose any 4 of the 7 clubs, each of the n students is the member of at least one of the chosen 4 clubs. However, for any of choice of 3 of the 7 clubs there is a student who is not a member of any of the three clubs. What is the smallest value of n?

SOLUTION. We first show that n is at least 35.

Label the clubs with $1, 2, \dots 7$. By assumption, for any choice $1 \leq a < b < c \leq 7$ there must be at least one student that is not a member of clubs a, b and c. For each student let's write down the list of triples a, b, c so that the student is not a member of any of these three clubs. Then we have at most one triple for each student: if we have two for student x then we would be able to find four distinct clubs so that x is not a member of any of them. This means that the number of triples is at most n. There are $\frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$ ordered triples we can choose from $1, 2, \dots, 7$: there are $7 \cdot 6 \cdot 5$ ways to choose three different numbers from $1, 2, \dots, 7$, and each ordered triple is represented in $3 \cdot 2 \cdot 1 = 6$ different ways among these choices. Hence we have $35 \leq n$.

We also have to show that $n = 35$ is possible, for this we show a construction of 7 clubs. Label the 35 triples $1 \leq a < b < c \leq 7$ with the numbers 1, 2, ...35, these will correspond to the 35 students. Now for each $1 \leq i \leq 7$ collect the numbers $1 \leq x \leq 35$ for which the triple corresponding to x does not contain the number i. This way we produced 7 subsets of $1, \dots, 35$ (these are the seven clubs). Suppose that there are 4 clubs (labeled a, b, c, d) and a student numbered x. Student x corresponds to a triple, so at least one of a, b, c, d is not an element of that triple. But then the club corresponding to that number will contain x, which is what we wanted to show.

5. A game is played on a 100×100 board. Ramon controls two white game pieces which start in the bottom left and top right corners. Josie controls two black game pieces which start in the bottom right and top left corners. The players move alternately. In each move, a player can move one of the game pieces under control to a vacant square which shares a common side with its current location. Ramon wins if he can move the two white game pieces to be next to each other (i.e., located in two squares with a common side) within his first 1000 moves; otherwise, Josie wins. Who has a winning strategy?

SOLUTION. (It was not clear from the statement of the problem who actually starts the game. The solution below is for the case when Ramon starts.)

We show that Josie can always prevent Ramon from having the two white game pieces next to each other, hence she has a winning strategy.

We say that the four game pieces are in a 'boxed' position if they form the four vertices of a rectangle (with sides parallel to the board's sides), and the white game pieces are in the bottom-left and top-right corners of this rectangle. At the beginning of the game the game pieces are in a boxed position. We will show that Josie has a strategy so that after her moves the game pieces are always in a boxed position. If this is the case then Ramon cannot win. If the rectangle is formed by the game pieces is 2×2 then the black pieces are blocked from moving next to each other by the white ones, and if the rectangle is larger than the black pieces are more then one steps away from each other.

Suppose that the game pieces are in a boxed position, and Ramon moves one of the black pieces. That piece had a white game piece in its row and another white piece in its column, moving the black piece, after the move one of these two white pieces will not share a row or a column. Moving that white piece in the same direction as Ramon's move will restore the boxed position.