WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET IV (2019-2020)

1. We are given a 2020-sided convex polygon. We want to select three distinct edges of the polygon, so that if we go around the edges in clockwise order, at least two unselected edges lie between every pair of selected edges. In how many different ways can we select the three edges?

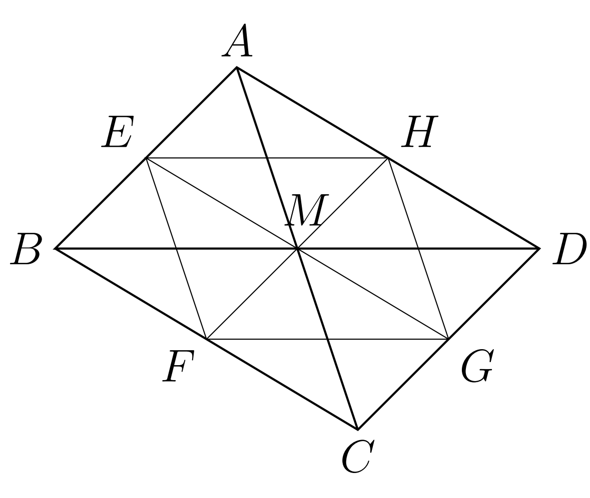
SOLUTION. Denote the edges of the polygon by $e_1, e_2, \dots, e_{2020}$ in a clockwise order. First we count the number of choices when e_1 is a chosen edge. In this case we cannot choose e_2, e_3, e_{2020} or e_{2019}. Thus we must pick two edges from the remaining 2015 edges $e_4, \dots, e_{2018}$, and these two edges must have at least two other edges between them. If the first chosen edge among these (in clockwise order) is e_k, then the second one can be $e_{k+3}, e_{k+4}, \dots, e_{2018}$. Thus we must have $4 \le k \le 2015$, and if we pick e_k as the first edge then we have $2018-(k+3)+1 = 2016-k$ choices for the remaining edge. (For e_4 we have 2012 choices, for e_5 we have 2011 choices, and so on.) Thus there are $2012 + \dots + 1 = \frac{2013\times 2012}{2}$ ways to choose three edges according to the instructions, with e_1 being one of the three.

Repeating this process for each edge e_k we would get $2020 \times \frac{2013\times2012}{2}$ choices for the three edges. However, this way we would count each selection three times (once for each edge). Hence, the actual number of choices is one third of this number: $\frac{2020\times2012\times2013}{6}$.

(The same argument shows that for a convex polygon with $n \ge 9$ sides the number of choices is $\frac{n(n-8)(n-7)}{6}$.)

2. On quadrilateral $ABCD$, points E, F, G, and H are the midpoints of sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, respectively. Suppose the diagonals $\overline{AC}$ and $\overline{BD}$ of quadrilateral $ABCD$ intersect at the same point as the diagonals $\overline{EG}$ and $\overline{FH}$ of quadrilateral $EFGH$. Show that $ABCD$ must be a parallelogram.

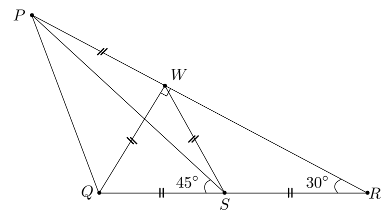
SOLUTION. Note that no matter what quadrilateral $ABCD$ we start with, the quadrilateral $EFGH$ connecting the midpoints of the sides of $ABCD$ is always a parallelogram. This is because $\overline{EF}$ and $\overline{GH}$ are midlines of $\triangle ABC$ and $\triangle ACD$, respectively, and, similarly, $\overline{EH}$ and $\overline{FG}$ are midlines of $\triangle ABD$ and $\triangle BCD$, respectively. This means that $\overline{EF} \parallel \overline{AC} \parallel \overline{GH}$ and $\overline{EH} \parallel \overline{BD} \parallel \overline{FG}$, so $EFGH$ is a parallelogram. Suppose that the diagonals of $ABCD$ and $EFGH$ all intersect at the same point M, as shown.



Because $\overline{EH}$ is the midline of $\triangle ABD$, it follows that the four triangles $\triangle AEH$, $\triangle EBM$, $\triangle HMD$, and $\triangle MHE$ are congruent. Similarly, $\triangle CGF$, $\triangle GDM$, $\triangle FMB$, and $\triangle MFG$ are congruent. Because $EFGH$ is a parallelogram, it follows that $\triangle MHE$ is congruent to $\triangle MFG$, so all eight of the previously mentioned triangles are congruent. In particular, $\angle MHD = \angle MFB$ implying that line FH crosses $\overline{BC}$ and $\overline{AD}$ at the same angles, and so $\overline{AD} \parallel \overline{CB}$. A symmetric argument shows that $\overline{AB} \parallel \overline{CD}$, and this proves that $ABCD$ is a parallelogram.

3. In triangle ΔPQR, the midpoint of side $\overline{QR}$ is denoted by S. Find $\angle QPR$, if we know that $\angle PRQ = 30°$ and $\angle PSQ = 45°$.

SOLUTION. In the triangle ΔSPR we have $\angle PSR = 180° - \angle PSQ = 135°$ and $\angle SRP = 30°$, hence we have $\angle SPR = 15°$.



Draw the altitude $\overline{QW}$ of $\triangle PQR$ to side $\overline{PR}$. Observe that ΔQWR is a right-angled triangle with S being the midpoint of the hypotenuse. Hence, we have $QS = SW = SR$. (This is the consequence of the converse of Thales's Theorem, which states that the midpoint of the hypotenuse of a right-angled triangle is equidistant from the three vertices of the triangle.) Furthermore, since $\angle PRQ = 30°$, we have $\angle SWQ = 90° - \angle SWR = 90° - \angle SRW = 60°$. The triangle ΔWQS is an isosceles triangle ($QS = WS$) with a $60°$ angle, which means that ΔWQS is an equilateral triangle. Using $\angle PSQ = 45°$, we conclude that $\angle WSP = 15°$. Thus, ΔWPS is isosceles (as the angles at P and S are equal to $15°$), leading to $WP = WS$. But we also have $WS = WQ$, and so $WP = WQ$. In other words, ΔWPQ is a isosceles-right triangle. This gives $\angle QPR = \angle WPQ = 45°$ and concludes the proof.

4. We construct a sequence of prime numbers $p_1, p_2, \ldots,$ as follows: We set $p_1 = 2$. For any $n \geq 1$, the integer p_{n+1} is the largest prime factor of the number which is one larger than the product of $p_1, \ldots, p_n$. (So for example, p_4 is the largest prime factor of $1 + p_1 \cdot p_2 \cdot p_3$.) Show that $p_n \neq 5$ for all $n \geq 1$.

SOLUTION. From the definition we have $p_2 = 3$. Observe that for $n \geq 2$, the product $p_1 \cdot p_2 \cdots p_n$ is divisible by 6. Hence, the number $p_1 \cdot p_2 \cdots p_n + 1$ cannot be divisible by 2 or 3. Thus, $p_n \neq 2, 3$ for any $n > 2$. This implies two facts: Firstly, 4 cannot be a divisor of $p_1 \cdots p_n$ (because $p_1 = 2$ is the only even factor in the product), and secondly, the only way to have $p_{n+1} = 5$ for some n is to get the equality $p_1 \cdot p_2 \cdots p_n + 1 = 5^k$ for some $k \geq 1$. (If 5 is the largest prime factor of $p_1 \cdot p_2 \cdots p_n + 1$, then it must be the only prime factor of that number.)

However 4 divides $5^k - 1$ because 5 has a remainder equal to 1 when divided by 4, so this is true for $5^k = 5 \cdots 5$ as well. But this means that if $p_1 \cdot p_2 \cdots p_n + 1 = 5^k$, then $p_1 \cdot p_2 \cdots p_n$ is divisible by 4, which is impossible. This shows that p_n can never be equal to 5.

5. The edge lengths of a triangle are given by $a, b,$ and c. We know that $ab + bc + ac = 12$. Show that the perimeter of the triangle cannot be larger than 7.

SOLUTION. The perimeter of the triangle is $a+b+c$. Note that

$$\begin{aligned}(a+b+c)^2=(a+b+c)(a+b+c)&=a^2+b^2+c^2+2(ab+bc+ac)\\&=a^2+b^2+c^2+24.\end{aligned}$$

On the other hand, the triangle inequality tells as that

$$a+b>c, \qquad a+c>b,$$

which implies

$$-a<b-c<a,$$

and

$$(b-c)^2<a^2.$$

The same way we get $(a-c)^2<b^2$ and $(a-b)^2<c^2$. Adding up these three inequalities we get

$$(a-b)^2+(b-c)^2+(a-c)^2<a^2+b^2+c^2.$$

But we also have

$$\begin{aligned}(a-b)^2+(b-c)^2+(a-c)^2&=a^2+b^2-2ab+b^2+c^2-2bc+a^2+c^2-2ac\\&=2(a^2+b^2+c^2)-24.\end{aligned}$$

This leads to the inequality

$$2(a^2+b^2+c^2)-24<a^2+b^2+c^2,$$

which gives

$$a^2+b^2+c^2<24.$$

Going back to our first identity we get

$$(a+b+c)^2=a^2+b^2+c^2+24<24+24<49=7^2.$$

Since a,b,c are all positive numbers, the inequality $(a+b+c)^2<7^2$ implies $a+b+c<7$.