WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET III (2019-2020)

1. Brianna has a map of the 48 contiguous states of the United States. (These are the states without Alaska and Hawaii.) She wants to color these states with three colors (green, yellow, or brown) so that each state is colored with one of these colors, and no state with the color green shares a border with a state that is colored yellow. Let N denote the number of such colorings. Prove that N is an odd number.

SOLUTION. Let P be a coloring pattern that Brianna can use. Let P' be the same pattern as P except that the states colored green in P are colored yellow in P' and those colored yellow in P are colored green in P'. Because no states colored green shares a border with a state colored yellow in P, the same holds for P' showing that P' is a valid way for Brianna to color the states. The pairing of each coloring pattern P with its corresponding coloring pattern P' pairs each valid coloring pattern with a different valid coloring pattern except for the one coloring pattern that colors all the states brown for which $P = P'$. This shows that all valid coloring patterns can be paired up except for one, and this implies that the number of such patterns must be odd.

2. The numbers $1, 2, \ldots, 2019$ are arranged in an arbitrary order $x_1, x_2, \ldots, x_{2019}$. (Here x_i denotes the ith number.) Show that there exist distinct indices i and j with $1 \le i < j \le 2019$ such that $|x_i - i| = |x_j - j|$

SOLUTION. Since the numbers $x_1, x_2, \ldots, x_{2019}$ are just $1, 2, \ldots, 2019$, they have the same sum:

$$x_1 + x_2 + \cdots + x_{2019} = 1 + 2 + \cdots + 2019,$$

which means that the sum of differences is zero:

$$(x_1 - 1) + (x_2 - 2) + \cdots + (x_{2019} - 2019) = 0$$

This means that an even number of the differences $x_i - i$ are odd, and the rest are even.

For any two integers a, b the difference $(a - b)$ has the same parity as $|a - b|$, so they are both even or they are both odd. Hence $x_i - i$ and $|x_i - i|$ always have the same parity, which means that among the expressions

$$|x_1 - 1|, |x_2 - 2|, \ldots, |x_{2019} - 2019|$$

we have an even number of odd values.

Observe that $|x_i - i|$ can take 2019 distinct values, since $0 \le |x_i - i| \le 2018$. If $|x_i - i| \ne |x_j - j|$ for any distinct i and j, then the numbers $|x_1 - 1|, |x_2 - 2|, \ldots, |x_{2019} - 2019|$ are just $0, 1, \ldots, 2018$ in a certain order. But the number of odd numbers among $0, 1, \ldots, 2018$ is odd, which leads to a contradiction. Thus, there are distinct i and j so that $|x_i - i| = |x_j - j|$.

3. A pentagon $A_1A_2A_3A_4A_5$ is inscribed in a circle of radius 10. The points $B_1, B_2, \ldots, B_5$ are chosen on the sides of the pentagon $\overline{A_1A_2}$, $\overline{A_2A_3}$, $\overline{A_3A_4}$, $\overline{A_4A_5}$, and $\overline{A_5A_1}$, respectively. Given that the *area* of the pentagon $A_1A_2A_3A_4A_5$ equals 200. Prove that the perimeter of the pentagon $B_1B_2 \ldots B_5$ is at least 40.

SOLUTION. Let O be the center of the circle. Note that O lies inside the pentagon $A_1A_2A_3A_4A_5$: otherwise, the pentagon would be entirely contained in a half-circle, but the area of the pentagon is 200, which is more than one half of the area of the circle (which is $\pi \cdot 10^2 \approx 314.15\ldots$).

Let us connect O to the vertices A_1, A_2, A_3, A_4, and A_5 of the pentagon $A_1A_2A_3A_4A_5$, and also to the vertices B_1, B_2, B_3, B_4, and B_5 of the pentagon $B_1B_2B_3B_4B_5$. Denote by A_1' the point of intersection of $\overline{OA_1}$ with the side $\overline{B_5B_1}$, by A_2' the point of the intersection of $\overline{OA_2}$ with the side $\overline{B_1B_2}$ and so on.

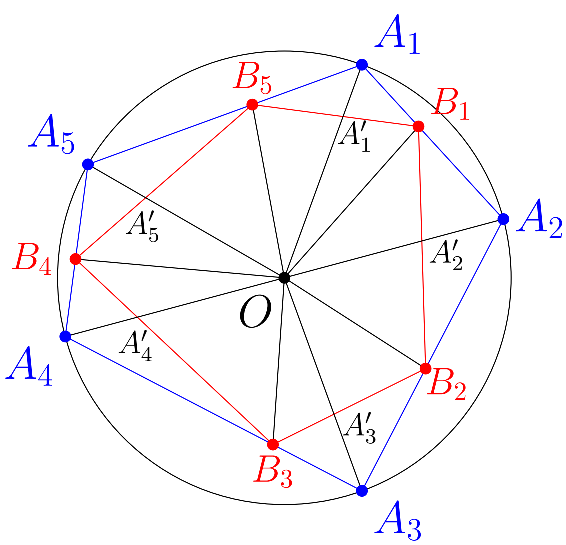
The pentagon $A_1A_2A_3A_4A_5$ is divided into ten triangles $\triangle OA_1B_1$, $\triangle OA_2B_1$, $\triangle OA_2B_2$, and so on. Consider the triangle $\triangle OA_1B_1$. Because the point A_1' lies on the side $\overline{OA_1}$, the distance B_1A_1' is more than or equal to the length of the altitude dropped from B_1 onto $\overline{OA_1}$, therefore, the area of the triangle $\triangle OA_1B_1$ is at most $\frac{1}{2} \cdot OA_1 \cdot B_1A_1'$. Since $\overline{OA_1}$ is a radius of the circle, its length is 10, therefore,

$$\text{Area of } \triangle OA_1B_1 \le 5 \cdot B_1A_1'.$$

Repeat this argument for all ten triangles and add the resulting inequalities. We see that

$$(\text{Area of } \triangle OA_1B_1)+(\text{Area of } \triangle OA_2B_1)+\cdots+(\text{Area of } \triangle OA_1B_5) \le 5\cdot(B_1A_1'+B_1A_2'+\cdots+B_5A_1').$$

It remains to notice that the ten triangles on the left side of the inequality form the pentagon $A_1A_2A_3A_4A_5$, so that their total area is 200, while the line segments on the right form the perimeter of the pentagon $B_1B_2B_3B_4B_5$.



4. For each number n, denote by a_n the integer closest to $\sqrt{n}$. That means that a_n is $\sqrt{n}$ rounded up or down in the usual way; for instance, $\sqrt{2} \approx 1.41\ldots$, so $a_2 = 1$, and $\sqrt{3} \approx 1.73\ldots$, so $a_3 = 2$. Find the sum $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{2019}}$.

SOLUTION. Let us fix an integer $k > 0$, and and find out for which n we have $a_n = k$. If $\sqrt{n}$ rounds to k, then $\sqrt{n}$ must satisfy

$$k - \frac{1}{2} \le \sqrt{n} < k + \frac{1}{2}.$$

Squaring the inequality, we see that this happens when

$$k^2 - k + 1/4 \le n < k^2 + k + 1/4.$$

As a side note, since n is an integer, both inequalities are in fact going to be strict. Thus $\sqrt{n}$ is never a half-integer (that is, it is never of the form $m + 1/2$ for an integer m), which means that we do not have to worry about whether to round half-integers up or down when we define a_n.

The number of n's satisfying this inequality is exactly $2k$. Thus, the sequence a_n is obtained by writing each integer k exactly $2k$ times: $1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, \ldots$. For $n = 2019$, $\sqrt{n} \approx 44.9\ldots$, thus $a_{2019} = 45$. Moreover, the smallest value of n for which $a_n = 45$ is $45^2 - 45 + 1 = 1981$. Thus, the sequence $a_1, a_2, a_3, \ldots, a_{2019}$ includes $2k$ instances of every integer k between 1 and 44, and exactly 39 instances of 45 (for all $n = 1980, \ldots, 2019$).

Taking the sum of the reciprocals, we obtain

$$2 \cdot \frac{1}{1} + 4 \cdot \frac{1}{2} + 6 \cdot \frac{1}{3} + \cdots + 88 \cdot \frac{1}{44} + 39 \cdot \frac{1}{45} = 2 + 2 + \cdots + 2 + \frac{39}{45},$$

with 2 appearing exactly 44 times on the right side of the equation. Therefore, the sum equals $2 \cdot 44 + \frac{39}{45} = 88\frac{39}{45} = 88\frac{13}{15}$.

5. We want to reach the number 2020 from the number 1 by performing a series of operations, In each step we can either triple our number or add 1. (E.g. we can triple 1 five times to get $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 729$ and add one 1291 times to arrive at 2020.) Find the minimal number of operations needed to get 2020 from 1. Make sure to prove that your answer gives the smallest number of steps.

SOLUTION. Imagine that you reverse the steps: in each step you can subtract 1 or divide 3. Now imagine that you divide by three whenever you can, otherwise you subtract 1. Starting from 2020 this gives:

$$2020 \xrightarrow{-1} 2019 \xrightarrow{\div 3} 673 \xrightarrow{-1} 672 \xrightarrow{\div 3} 224 \xrightarrow{-1} 223 \xrightarrow{-1} 222 \xrightarrow{\div 3}$$
$$74 \xrightarrow{-1} 73 \xrightarrow{-1} 72 \xrightarrow{\div 3} 24 \xrightarrow{\div 3} 8 \xrightarrow{-1} 7 \xrightarrow{-1} 6 \xrightarrow{\div 3} 2 \xrightarrow{-1} 1$$

Reversing the steps we find 15 steps will take us from 1 to 2020.

To show that we cannot do that with fewer than 15 steps requires more work. Let $f(n)$ denote the sum of the number of digits and the sum of the digits in the base three representation of n. For example, 15 can be written in base 3 as 120 (because $15 = 9 + 2 \cdot 3 + 0 \cdot 1$), so $f(15) = 3 + 3 = 6$ (the length is 3 and the sum of the digits is $2 + 1 = 3$). 31 is written as 1011 in base 3, so $f(32) = 4 + 3 = 7$ (the length is 4, and the sum of the digits is 3).

We will show, that if we apply any of the allowed steps to n, then $f(n)$ can increase by at most one. Since 2020 is 2202211 in base 3, that means that $f(2020) = 7 + 10 = 17$. Since $f(1) = 1 + 1 = 2$, this shows that we need at least 15 steps to go from 1 to 2020.

When we multiply a number by three, we just add an extra 0 digit at the end of its base three representation. Hence $f(3n) = f(n) + 1$. If we add 1 to n, then the following things can happen:
Case 1: the last digit of n (in base 3) is 0 or 1. In that case the last digit increases by one, but the length does not change, so $f(n + 1) = f(n) + 1$.
Case 2: the last digit (but not all digits) are equal to 2. In that case there is a maximal block of 2s at the end of the number. (E.g. for 1101222 this is the block 222.) When we add 1, the block of 2s at the end will change to a block of 0s, and the last digit that is not equal to 2 will increase by one. In that case the length will not change, and the sum of the digits decrease, so $f(n + 1) \leq f(n)$.
Case 3: the number n is just built from 2s. In that case adding one will increase the length by one, but the sum of the digits will decrease (since the digits of $n + 1$ will be 1 followed by zeros).

Hence $f(n+1) \le f(n)$ in this case as well.

This shows that $f(n)$ can only increase by at most one in each step. Hence we cannot reach 2020 from 1 with fewer than 15 steps.