WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET II (2019-2020)

1. Is there an integer n so that the sum of its digits is equal to the sum of the digits of n^2, and the latter sum exceeds 2019?

SOLUTION. Let's call an integer *nice* if the sum of its digits is equal to the sum of the digits of its square. With a little bit of experimenting, we can check that up to 100, these are the nice numbers: 1, 9, 10, 18, 19, 45, 46, 55, 90, 99, 100. We can also check that 999 is nice, as $999^2 = 998001$. This suggests to try out the numbers that are formed from k consecutive nines: $9 \dots 9 = 10^k - 1$. The sum of the digits of this number is $9k$. The square is given by

$$(10^k - 1)^2 = 10^{2k} - 2 \cdot 10^k + 1 = 1\underbrace{00\dots0}_{2k \text{ zeros}} - 2\underbrace{00\dots0}_{k \text{ zeros}} + 1.$$

Evaluating the expression at the right, we get

$$(10^k - 1)^2 = \underbrace{9 \cdots 9}_{k-1 \text{ nines}} 8 \underbrace{0 \cdots 0}_{k-1 \text{ zeros}} 1,$$

and thus, the sum of the digits of the square is $9(k-1) + 8 + 1 = 9k$, which is the same as the sum of the digits of the original number. By choosing k so that $9k \geq 2019$ (e.g., with $k = 225$), we get a number that satisfies the requirements.

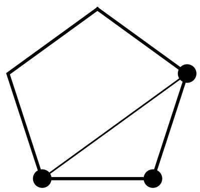
2. Show that for any positive integer k, there is a perfect square among the $k+1$ consecutive integers $k, k+1, \dots, 2k$.

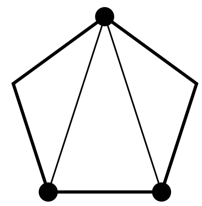
SOLUTION. Note that 1^2 appears among the numbers $1, 2$, while 2^2 appears among the numbers $2, 3, 4$, the numbers $3, 4, 5, 6$, and the numbers $4, 5, 6, 7, 8$. Thus, the statement is true for $k \leq 4$. Suppose that for some $k \geq 5$, the statement is not true. This would mean that there is a positive integer m with $m^2 < k$ and $(m+1)^2 > 2k$. But then $2m^2 < 2k < (m+1)^2 = m^2 + 2m + 1$, which implies that $m^2 < 2m + 1$ and $m^2 - 2m + 1 < 2$. This means $(m-1)^2 < 2$, so $m \leq 2$ and $2k < (m+1)^2 \leq 9$. This contradicts $k \geq 5$, so the result must be true for all k.

The statement can also be shown by induction on k. The result is true for $k = 1, 2, 3$, and 4, as stated above. Assume that it is true for a particular $k > 3$. That means that there is a perfect square among $k, k+1, k+2, \dots, 2k$. If k is a perfect square, say $k = m^2$, then m must be at least 2. This shows that $(m+1)^2 = m^2 + 2m + 1 = k + 2m + 1 \leq k + m \cdot m + 1 = 2k + 1$, so the perfect square $(m+1)^2$ is among $k+1, k+2, k+3, \dots, 2(k+1)$. Alternatively, if k is not a perfect square, then one of $k+1, k+2, k+3, \dots, 2k$ is a perfect square, and, again, there is a perfect square among $k+1, k+2, k+3, \dots, 2(k+1)$. In either case, the statement is true for $k+1$, which completes the proof by induction.

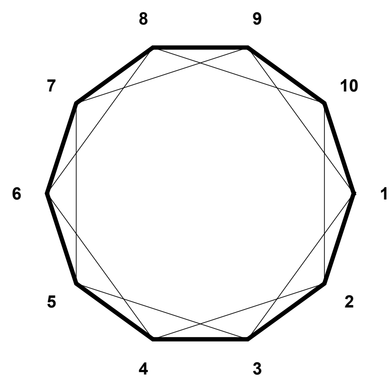
3. A regular decagon (ten-sided polygon) has five arbitrarily chosen vertices colored red. Show there exists an isosceles triangle whose vertices are three of these five red vertices.

SOLUTION. We first show that in a regular pentagon, any three of the vertices form a triangle that is isosceles. First note that choosing three vertices of the pentagon is equivalent to not choosing two vertices. These two "unchosen" vertices are either next to each other, or there is a chosen vertex between them. Hence, the three chosen vertices form a triangle as shown in one of the two pictures below. In the first case, the triangle is isosceles, as two of the sides have the same length (the edge length of the pentagon). In the second case, the two sides of the triangle which are diagonals of the pentagon have the same length (this follows by symmetry), so we get an isosceles triangle in this case, as well.





Label the vertices of the decagon $1, 2, \ldots, 10$, in clockwise order. Observe that vertices $1, 3, 5, 7, 9$ form a regular pentagon, and vertices $2, 4, 6, 8, 10$ also form a regular pentagon.



Call these pentagons P_1 and P_2. Since we are coloring five of the ten vertices red, at least one of P_1 or P_2 must contain at least three of these red vertices. But any three of these red vertices will be the vertices of an isosceles triangle.

4. 2019 rocks, all of different weights, are placed in a circle. Between each adjacent pair of rocks lies a ball with weight equal to the difference between the weights of the two rocks. Show that the 2019 balls can be distributed among the two pans of a balance scale so that the pans perfectly balance.

SOLUTION.

Let us denote the weights of the rocks by $r_1, r_2, \ldots, r_{2019}$. The ball that is between rocks k and $k+1$ weighs $r_k - r_{k+1}$ if $r_k > r_{k+1}$ and $r_{k+1} - r_k$ otherwise. (In other words, the weight is the absolute value $|r_k - r_{k+1}|$.) Of course, the ball between rock 2019 and rock 1 weighs $|r_{2019} - r_1|$.

As we distribute the rocks between the two pans, let us keep track of the difference D between the total weight on the left pan and the total weight on the right. Here are the possibilities:

If $r_k > r_{k+1}$, the ball between the two rocks weighs $r_k - r_{k+1}$. We have two choices:

- Putting this ball on the left will increase the weight on the left by $r_k - r_{k+1}$ and change the difference D by $r_k - r_{k+1}$;

- Putting this ball on the right will increase the weight on the right by $r_k - r_{k+1}$ and change the difference D by $r_{k+1} - r_k$.

On the other hand, if $r_{k+1} > r_k$, the ball between the two rocks weighs $r_{k+1} - r_k$, and we have two choices again:

- Putting this ball on the left will increase the weight on the left by $r_{k+1} - r_k$ and change the difference D by $r_{k+1} - r_k$;
- Putting this ball on the right will increase the weight on the right by $r_{k+1} - r_k$ and change the difference D by $r_k - r_{k+1}$.

We now see that regardless of which rock is heavier, we have a choice between changing D by $r_k - r_{k+1}$ or by $r_{k+1} - r_k$. Our goal is to have $D = 0$ (the scales are in balance, as all things should be) after all rocks are placed. This can be accomplished by having the difference equal to

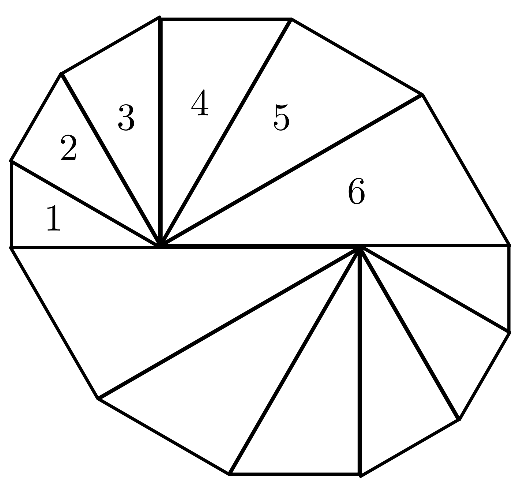
$$D = (r_2 - r_1) + (r_3 - r_2) + (r_4 - r_3) + \cdots + (r_{2019} - r_{2018}) + (r_1 - r_{2019}) = 0.$$

The fact that the expression simplifies to zero can be checked by noting that each term r_k shows up with a plus sign and a minus sign, hence the contributions will all cancel out.

This leads to the following solution: If the $(k+1)$-st rock is heavier than the k-th, we put the ball between them on the left pan of the scales; otherwise, we put it on the right. Similarly, if the first rock is heavier than the last, we put the ball between them on the left pan; otherwise, we put it on the right.

5. We know that a convex polygon can be divided into $30° - 60° - 90°$ triangles which are not necessarily congruent to each other. What is the maximum number of sides such a polygon can have? (You must provide a construction to prove that such a polygon with this number of sides exists, and also show that this is the maximal such number.)

SOLUTION. When we divide the polygon into $30° - 60° - 90°$ triangles, each interior angle of the polygon is the sum of angles from these triangles. Hence, each interior angle of the polygon must be an integer multiple of $30°$. Since the polygon is convex, this means that each interior angle must be $30°$, $60°$, $90°$, $120°$, or $150°$. Thus each exterior angle is at least $180° - 150° = 30°$. Since the sum of the exterior angles of convex polygon is equal to $360°$, the number of sides is therefore bounded by $\frac{360°}{30°} = 12$.



We now show that we can construct a convex polygon with 12 sides satisfying the desired property. Note that in order to exactly achieve the upper bound, all exterior angles must be equal to $30°$, so all interior angles must be equal to $150°$. The construction proceeds as follows: First take a $30° - 60° - 90°$ triangle (marked 1 in the figure). Then take a slightly larger rotated copy of the triangle (marked 2 in the figure) and paste it to triangle 1, so that the two triangles share a side. Note that the interior angle formed from the two triangles is $60° + 90° = 150°$. Continue pasting triangles 3, 4, 5, and 6 in a radial manner. This produces five interior angles that all measure $150°$. Furthermore, the sides of triangles 1 and 6 form a straight line, since the sum of the interior angles of the six triangles is $30° \times 6 = 180°$.

To complete the construction, we simply flip the heptagon formed by the first six triangles and paste the flipped heptagon together with the original heptagon. Note that the interior angles formed by pasting together the heptagons also measure $90° + 60° = 150°$, as we want. Thus, we have constructed a convex 14-gon using 12 different $30° - 60° - 90°$ triangles.