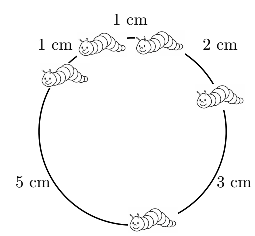
WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH
SOLUTIONS TO PROBLEM SET I (2019-2020)

1. Seven friends who share an apartment also share a dog. They want to create a schedule so that each of the friends takes care of the dog on one day of the week. Each of the friends submits a list of the days they are willing to care for the dog. Each list has a different number of days from 1 to 7. Show that the friends can always create a schedule where each one takes care of the dog on a day on their list.

SOLUTION. Because there are seven friends, and each list has a different number of days, there is a friend whose list has 1 day, one whose list has 2 days, and so forth for 3 days, 4 days, 5 days, 6 days, and 7 days. For the friend with only 1 day on their list, assign them to care for the dog on the day they listed. For the friend with only 2 days listed, at least one of their listed days was not already assigned, so pick one of those days not already assigned and assign that day to this friend. In general, suppose we have assigned k days to the friends whose lists contain $1, 2, 3, \ldots,$ and k days, where $1 < k < 7$. There is one friend with $k+1$ days on their list. At least one of those days has not yet been assigned, so pick one day on their list that has not been assigned, and assign it to that friend. Repeating this procedure for $k = 2, 3, \ldots, 6$ will give us an assignment where all seven days are assigned to all seven friends, and no friend is assigned more than one day.

2. Five caterpillars are sitting around the rim of a circular leaf, so that successive distances between neighboring caterpillars (measured along the perimeter) are 1, 1, 2, 3, and 5 cm. The caterpillars would like to conduct a meeting, which requires them to crawl around the rim of the leaf to a single predetermined point. What is the optimal location of the meeting, which minimizes the total distance required by all five caterpillars to travel?



SOLUTION. Consider unwrapping the edge of the circular leaf with respect to a potential meeting point, so that the initial locations of the caterpillars correspond to distinct points in the interval $[0, 12]$, and the meeting point is at 6. (For example, if the caterpillars convene their meeting directly between the first two caterpillars, the unwrapped interval will have caterpillars situated at the points $\{0.5, 5.5, 6.5, 7.5, 9.5\}$.) We claim that if none of the caterpillars are sitting at the point 6, the meeting point *cannot* be optimal. Indeed, all caterpillars in the interval $[0, 6]$ should crawl to the right in order to attend the meeting, whereas the caterpillars in the interval $[6, 12]$ should crawl to the left in order to attend the meeting. Furthermore, if no caterpillar is sitting at 6, one interval (e.g. the interval $[0, 6]$) must contain more caterpillars than the other. In particular, moving the meeting location to the left by any small number $\epsilon > 0$ will *decrease* the total distance required by the caterpillars to travel, since each caterpillar in $[6, 12]$ has its travel distance increased by at most ϵ, whereas each caterpillar in $[0, 6]$ has its travel distance decreased by ϵ.

Thus, we conclude that the meeting point must coincide with one of the initial locations of the five caterpillars. It is easy to check that the total travel distance required if the meeting is held at the locations of the successive caterpillars is:

$$\begin{aligned} 0 + 1 + 2 + 4 + 5 &= 12, \\ 1 + 0 + 1 + 3 + 6 &= 11, \\ 2 + 1 + 0 + 3 + 4 &= 10, \\ 4 + 3 + 2 + 0 + 3 &= 12, \\ 5 + 6 + 5 + 3 + 0 &= 19, \end{aligned}$$

so the optimal location of the meeting is at the initial location of the third caterpillar.

3. Are there 6 natural numbers such that none of them is divisible by any of the others, but the square of each is divisible by all of the others?

SOLUTION. The problem becomes easier if we think about numbers in terms of their prime factorization. If a and b are two numbers, we see that a does not divide b if some prime occurs in a higher power ("multiplicity") in the factorization of a than in b; for example, $4 = 2^2$ does not divide $6 = 2 \cdot 3$ because the prime 2 occurs twice in the factorization of 4 and only once in the factorization of 6. Similarly, if we want a to divide b^2, we need to make sure that every prime that occurs in the factorization of a occurs in the factorization of b at least half as many times: then it will occur in b^2 at least as many times as in a, ensuring that a divides b^2. For instance, continuing the above example, 4 divides 6^2.

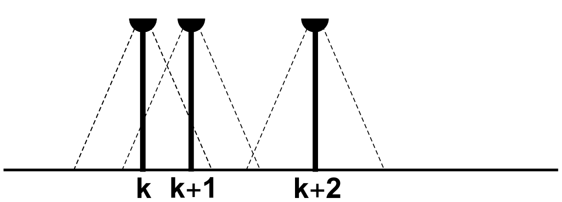
The easiest (but not the only) way to come up with the six numbers with the required property is to use six primes (say, 2, 3, 5, 7, 11, and 13), and let each number have the factorization containing one of the primes twice and the rest only once, giving the following six numbers:

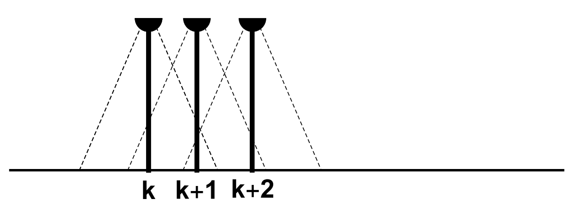
$$2^2{\cdot}3{\cdot}5{\cdot}7{\cdot}11{\cdot}13, \quad 2{\cdot}3^2{\cdot}5{\cdot}7{\cdot}11{\cdot}13, \quad 2{\cdot}3{\cdot}5^2{\cdot}7{\cdot}11{\cdot}13, \quad 2{\cdot}3{\cdot}5{\cdot}7^2{\cdot}11{\cdot}13, \quad 2{\cdot}3{\cdot}5{\cdot}7{\cdot}11^2{\cdot}13, \quad 2{\cdot}3{\cdot}5{\cdot}7{\cdot}11{\cdot}13^2.$$

None of these numbers divides any of the others because each one contains the square of one of the primes in the factorization, while the rest only contain the prime itself. (In fact, the ratio of any two numbers in the list is a fraction whose numerator and denominator are different prime numbers.) On the other hand, the square of any of these numbers will be divisible by any of the other numbers because the square will contain all six of the primes with multiplicity at least two (one of them, in fact, will have multiplicity four).

4. There are N lights along a one-mile long straight path, not necessarily equally spaced. Each (dim) light illuminates a two-yard long stretch of the path: one yard both ways counted from the position of the lamp. Somebody noticed that the the entire path is illuminated, but if any light goes out, this will no longer be true. What is the largest possible value of N? (1 mile is 1760 yards.)

SOLUTION. Let us number lights from the start of the path in order of their location. Then the stretch lit by any light (let us say, the light number k) must overlap with, or at least be adjacent to, the stretch lit by the next light (number $k+1$), otherwise, there will be a dark zone between the two lights. However, the stretch lit by light k must not overlap with the stretch lit by light $k+2$, otherwise, the light between them (number $k+1$) could go out, and the path would remain illuminated. (As in the case of the picture on the right.)





In other words, the distance between the light number k and the light number $k+1$ should not exceed 2 yards, while the distance between light k and light number $k+2$ must exceed 2 yards. Thus, the third light is at least 2 yards from the first, the fifth is at least 2 yards further down the path and so on. This shows that it is impossible to have 1761 or more lights: the distance between the first one and the 1761st light would have to exceed one mile.

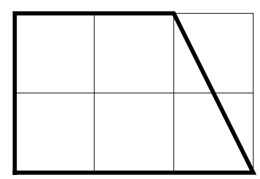
On the other hand, it is possible to place exactly 1760 lights according to the conditions. Let's place the first one 1/4 yard from the beginning of the path and the last one 1/4 yard from the end of the path. The remaining 1758 lights are then placed between these at equal distances of $\frac{1759.5}{1759}$. Then every point on the path is within a yard of at least one light, so the whole path is illuminated. But the location of a light is not illuminated by any of the other lights (since the distance between neighboring lights is larger than 1), hence if we turn off a light then the path cannot stay fully illuminated.

Remark: The problem could also be interpreted in a way so that each light *has to* illuminate exactly a two-yard stretch. Then the first lamp must be placed at position 1 and the last one must be placed at position 1759, which means that their distance is exactly 1758. By our previous arguments the distance between light 1 and light $2k+1$ must exceed $2k$, thus we cannot have 1759 or more lights. We can have 1758 lights satisfying the conditions for example by placing 879 lamps starting at 1 with $2+\frac{1}{2000}$ yards apart:

$$1, 3+\frac{1}{2000}, 5+\frac{2}{2000}, \dots, 1757+\frac{878}{2000},$$

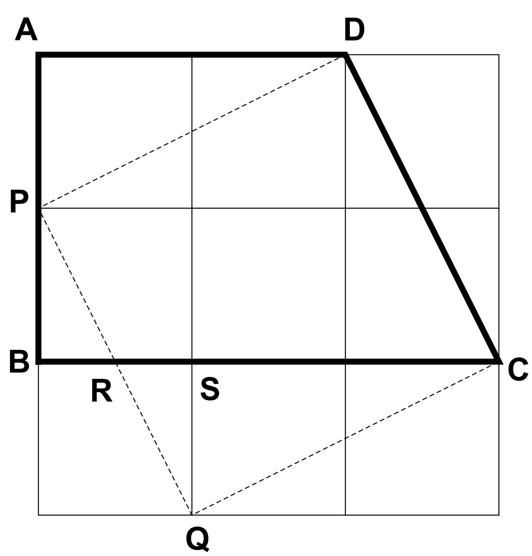
and then placing 879 lamps starting at 1759 with $2+\frac{1}{2000}$ yards apart in the opposite direction:

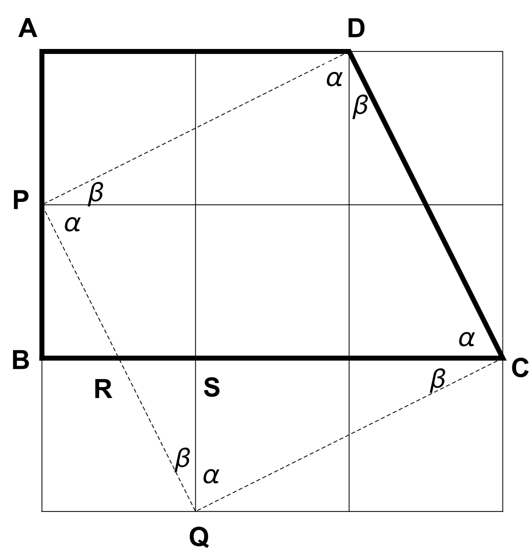
$$1759, 1757-\frac{1}{2000}, 1755-\frac{2}{2000}, \dots, 2-\frac{878}{2000}.$$



5. The trapezoid on the right was drawn in a 2×3 grid. Is it possible to cut this trapezoid into three parts so that the parts can be reassembled into a square?

SOLUTION. Extend the grid with an extra row to make it 3×3, as shown below. We label the vertices of the trapezoid by A, B, C, and D as shown. P, Q, and S are points on the grid as shown below. R is the intersection of PQ and BC.





We will prove that the quadrilateral $PQCD$ is a square. Each side of this quadrilateral is the hypothenuse of a right angle triangle with legs that are 1 and 2 unit long, which means that the four sides are equal. (We can even compute the length to be $\sqrt{1^2 + 2^2} = \sqrt{5}$ using the Pythagorean Theorem, but that's not important for us.) We can also see that each angle of $PQCD$ is the sum of the two acute angles in the same right angle triangle (denoted by α and β in the picture). But the acute angles in a right angle triangle add up to 90^o, which proves that $PQCD$ is a square.

Next we show that if we cut the trapezoid along the lines PD and PR, then the resulting three parts can be rearranged to form the square $PQCD$. The triangle APD is congruent to the triangle SQC as they are both right angle triangles with legs 1 and 2. The right angled triangles PBR and RQS are also congruent as their corresponding angles are the same, and we also have $PB = SQ$. This means that if we cut the trapezoid $ABCD$ along the lines PD and PR, then we can move the triangle APD into the triangle SQC and the triangle PBR into the triangle RQS, producing the square $PQCR$.