WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET V (2018-2019)

1. There are 100 special points in space, with no three special points on the same line. Show that we can always draw 2019 line segments, each of which has special points as endpoints, in a way so that we do not draw any triangle with three special points as vertices.

SOLUTION. We color any 50 of the special points red and the other 50 blue. There are $2500 = 50 \times 50$ line segments with one blue endpoint and one red endpoint. We draw any 2019 of these segments. We will have not drawn any triangle with three special vertices, because among any three special points, at least two are the same color, and we have not drawn any segments between two special points of the same color. (Since no three special points are on the same line, we cannot have drawn a segment between two points of the same color incidentally as we drew our segments above.)

2. We call a positive integer balanced if its decimal digits can be divided into two groups so that the sums of these groups are equal. (E.g. 22, 101, 134 are all balanced.) Find the smallest positive integer n so that both n and $n+1$ are balanced.

SOLUTION.

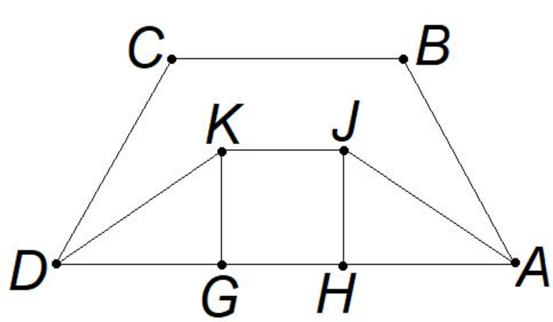
For any positive integer i, denote the sum of its digits as $S(i)$. If i is balanced then $S(i)$ must be an even number.

Let n be the smallest positive integer such that n and $n+1$ are both balanced. This means that $S(n)$ and $S(n+1)$ must both be even. If the units place digit of n is not equal to 9, then increasing it by one will only increase its unit digit by one, which means that $S(n+1) = S(n)+1$. In this case it is not possible for $S(n)$ and $S(n+1)$ to be simultaneously even. This implies the units place digit of n must be 9, and that of $n+1$ must be 0. It is easy to verify that $n+1$ cannot be a 2-digit number, since no 2-digit positive integer ending in 0 is balanced. If $n+1$ is a 3-digit number, then it must be of the form $xx0$ where $1 \le x \le 9$, as these are the only 3-digit balanced numbers ending in 0. As x ranges from 1 to 9, we set $n+1 = xx0$ and check if $n = xx0 - 1$ is balanced. The smallest x for which this holds is $x = 5$, and so we conclude that $n = 549$.

3. Inside the regular hexagon $ABCDEF$, color points red if they are closer to the diagonal $\overline{AD}$ than to the outside of the hexagon, and otherwise, color the points blue. Find the ratio of the area of the red region to the area of the blue region.

SOLUTION. Let diagonal $\overline{AD}$ be drawn horizontally. It is sufficient to consider the half of the hexagon lying above the diagonal as shown below. Let K and J be the points on the bisectors of $\angle ADC$ and $\angle BAD$, respectively, such that K and J are half way between lines AD and BC. Let G and H be the points on AD for which KG and JH are perpendicular to AD. A point is closer to the diagonal $\overline{AD}$ than to side $\overline{AB}$ if it lies below line AJ, the bisector of $\angle BAD$. Similarly, a point is closer to $\overline{AD}$ than to side $\overline{CD}$ if it lies below line DK, the bisector of $\angle CDA$. Finally, a point is closer to $\overline{AD}$ than to side $\overline{BC}$ if it lies below line JK half way between the parallel lines AD and BC.

Without loss of generality, let the hexagon have side length 1, so $AD = 2$. Because $\angle ADC = \angle BAD = 60°$, the distance from B and C to line AD is $\frac{\sqrt{3}}{2}$. Thus, $GK = HJ = \frac{\sqrt{3}}{4}$. Because



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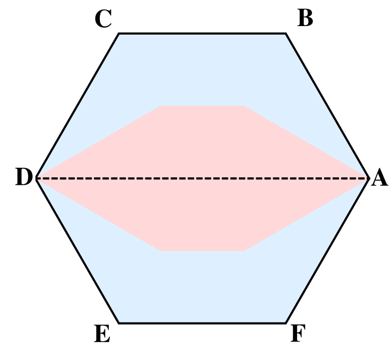


$\angle KDG = 30°$, it follows that $DG = \sqrt{3} \cdot \frac{\sqrt{3}}{4} = \frac{3}{4}$. This implies that $KJ = GH = 2 - 2 \cdot \frac{3}{4} = \frac{1}{2}$. The region colored red is then the trapezoid $AJKD$, which has area

$$\frac{2 + \frac{1}{2}}{2} \cdot \frac{\sqrt{3}}{4} = \frac{5\sqrt{3}}{16}.$$

Because trapezoid $ABCD$ has area

$$\frac{1+2}{2} \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{4},$$

the requested ratio of red area to blue area is

$$\frac{\frac{5\sqrt{3}}{16}}{\frac{3\sqrt{3}}{4} - \frac{5\sqrt{3}}{16}} = \frac{5}{7}.$$

4. The points $A_1, A_2, \ldots, A_{2019}$ are in the plane and they are not all on the same line. We know that there are two different points P and Q so that

$$A_1P + \cdots + A_{2019}P = A_1Q + \cdots + A_{2019}Q.$$

Show that there must be a point R with

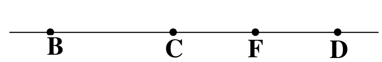
$$A_1R + \cdots + A_{2019}R < A_1P + \cdots + A_{2019}P.$$

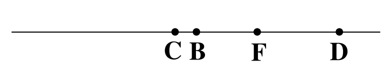
SOLUTION. We show that if R is the midpoint of $\overline{PQ}$ then

$$A_1R + \cdots + A_{2019}R < A_1P + \cdots + A_{2019}P.$$

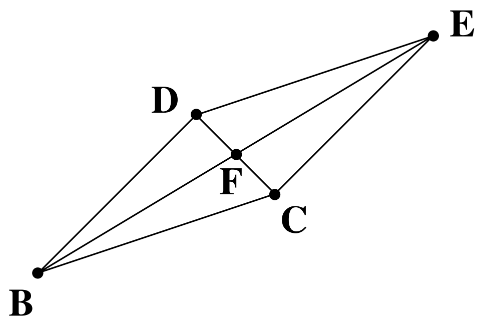
We first show that if B, C, D are any points and F is the midpoint of $\overline{CD}$, then $2BF \le BC + BD$. Moreover, if B, C, D are not on the same line then $2BF \le BC + BD$.

If B, C, D are on the same line and B is not on the line segment $\overline{CD}$, then $BF = \frac{1}{2}(BC + BD)$. If B, C, D are on the same line and B is on the line segment $\overline{CD}$, then $BF \le \frac{1}{2}CD = \frac{1}{2}(BC + BD)$.





Finally, if B, C, D are not on the same line, then extend the triangle BCD into a parallelogram $BCED$ by rotating B 180 degrees about the point F. Then $BF = FE$, and hence $BF = \frac{1}{2}BE$. In the triangle BCE we have $BE < BC + CE = BC + BD$, where we also used that the parallel sides of the parallelogram have the same length. This gives $2BF < BC + BD$.



Getting back to our original problem, we use the previous statement for the points A_j, P, Q and the midpoint R of PQ for all $1 \le j \le 2019$. This gives the inequalities $A_jR \le \frac{1}{2}(A_jP + A_jQ)$, and if we add them up, we get

$$A_1R + \cdots + A_{2019}R \le \frac{1}{2}\left(A_1P + \cdots + A_{2019}P + A_1Q + \cdots + A_{2019}Q\right) = A_1P + \cdots + A_{2019}P, \qquad (*)$$

since $A_1P + \cdots + A_{2019}P = A_1Q + \cdots + A_{2019}Q$. For given $1 \le j \le 2019$ the points A_j, P, Q are on the same line if A_j is on the line PQ. If this holds for all $j = 1, \ldots, 2019$, then the points $A_1, \ldots, A_{2019}$ would be on the same line. But this contradicts the assumption of the problem, which means that there must be at least one index j so that A_j, P, Q are not on the same line. Then the corresponding inequality is a strict inequality: $A_jR < \frac{1}{2}(A_jP + A_jQ)$ which means that we have strict inequality in the final inequality $(*)$ as well.

5. We write down the numbers $(44 + \sqrt{2019})^n$ for $n = 1, 2, 3, \ldots$ in order, and for each number we delete all the decimal digits after the decimal points. Show that in the resulting sequence of integers the even and odd numbers alternate.

SOLUTION. When we expand all the parentheses in the expression $(44 + \sqrt{2019})^n$ we get a sum where each term is produced by picking either 44 or $\sqrt{2019}$ from each of the n terms of $44 + \sqrt{2019}$ and multiplying these together. (E.g. for $n = 3$ one of the terms is $44 \cdot \sqrt{2019} \cdot 44$, but there are seven others.) That means that each term in the sum will be of the form $44^k(\sqrt{2019})^{n-k}$ where k can be any integer between 0 and n. When we expand the expression $(44 - \sqrt{2019})^n$ we will have a similar sum, but now each term is produced by picking either 44 or $-\sqrt{2019}$ from each of the n terms of $44 - \sqrt{2019}$ and multiplying these together. Hence here each term will be of the form $44^k(-\sqrt{2019})^{n-k}$.

The terms in the expanded versions of $(44 + \sqrt{2019})^n$ and $(44 - \sqrt{2019})^n$ can be matched up by making the same choices in the corresponding terms. That means that whenever we pick 44 in the first product, we choose 44 in the second one as well, and whenever we choose $\sqrt{2019}$ in the first case we will choose $-\sqrt{2019}$ in the second. A term of the form $44^k(\sqrt{2019})^{n-k}$ from the first sum is matched up with a term of the form $44^k(-\sqrt{2019})^{n-k}$. If $n - k$ is odd then $44^k(-\sqrt{2019})^{n-k} = -44^k(\sqrt{2019})^{n-k}$, so the sum of the matched terms will be zero. If $n - k$ is even then

$$44^k(-\sqrt{2019})^{n-k} = 44^k(\sqrt{2019})^{n-k} = 44^k 2019^{\frac{n-k}{2}}$$

which means that the two matched terms are equal, and they are integers, so their sum will be an even number. But this means that the sum $a_n = (44 + \sqrt{2019})^n + (44 - \sqrt{2019})^n$ is an even number because the matched terms in their expanded form always add up to an even integer.

Since $44^2 = 1936$ and $45^2 = 2025$, we have that $44 \leq \sqrt{2019} \leq 45$, and thus $44 - \sqrt{2019}$ is negative and $|44 - \sqrt{2019}| < 1$. So $(44 - \sqrt{2019})^n$ is positive for even n and negative for odd n, and always less than 1 in absolute value. This means that if n is even then the integer number $a_n = (44 + \sqrt{2019})^n + (44 - \sqrt{2019})^n$ is equal to $(44 + \sqrt{2019})^n$ plus a positive number that is less than 1. This means that we get $a_n - 1$ if we delete all the decimal digits after the decimal point in $(44 + \sqrt{2019})^n$. Since a_n is even, the number we get this way is odd.

Similarly, if n is odd then the integer number $a_n = (44 + \sqrt{2019})^n + (44 - \sqrt{2019})^n$ is equal to $(44 + \sqrt{2019})^n$ plus a negative number that is less than 1 in absolute value. This means that we get a_n if we delete all the decimal digits after the decimal point in $(44 + \sqrt{2019})^n$. Since a_n is even, this means that in this case we get an even number.

This shows that the even and odd numbers in our sequence will alternate and the sequence starts with an even number.

Here is another way to show that a_n is always an even integer. If we expand $(44 + \sqrt{2019})^n$ and $(44 - \sqrt{2019})^n$ for some small values ($n = 1, 2, 3$) we see that $(44 + \sqrt{2019})^n = x_n + y_n\sqrt{2019}$ and $(44 - \sqrt{2019})^n = x_n - y_n\sqrt{2019}$ with some integers x_n, y_n. To show that this pattern holds for all n we can use induction. For $n = 1$ the statement is true with $x_1 = 44$ and $y_1 = 1$. Assuming that we have

$$(44 + \sqrt{2019})^n = x_n + y_n\sqrt{2019}, \qquad (44 - \sqrt{2019})^n = x_n - y_n\sqrt{2019}$$

for a given positive integer n we also have

$$\begin{aligned}
(44 + \sqrt{2019})^{n+1} &= (x_n + y_n\sqrt{2019})(44 + \sqrt{2019}) = 44x_n + 2019y_n + (44y_n + x_n)\sqrt{2019} \\
(44 - \sqrt{2019})^{n+1} &= (x_n - y_n\sqrt{2019})(44 - \sqrt{2019}) = 44x_n + 2019y_n - (44y_n + x_n)\sqrt{2019}.
\end{aligned}$$

This shows that with the integer values $x_{n+1} = 44x_n + 2019y_n$ and $y_{n+1} = 44y_n + x_n$ the statement also holds for $n + 1$, and hence for all n. But then

$$a_n = (44 + \sqrt{2019})^n + (44 - \sqrt{2019})^n = x_n + y_n\sqrt{2019} + x_n - y_n\sqrt{2019} = 2x_n$$

which is indeed always an even integer. From this point the proof can be completed the same way as in the previous case.