WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET IV (2018-2019)

1. How many 5-digit integers $\underline{ABCDE}$ can we find so that $\underline{AB}, \underline{BC}, \underline{CD}, \underline{DE}$ are all two-digit squares? Here $\underline{A}, \underline{B}, \underline{C}, \underline{D}, \underline{E}$ denote digits, and they are not necessarily different.

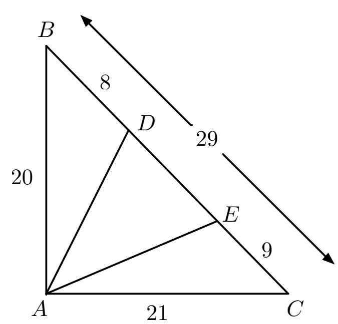
SOLUTION. The two digit squares are $16, 25, 36, 49, 64, 81$. We write $\underline{A} \to \underline{B}$ if $\underline{A}, \underline{B}$ is a two-digit square, and have the following.

$$\begin{tikzcd} 8 \arrow[r] & 1 \arrow[r] & 6 \arrow[r] & 4 \arrow[r] & 9 \\ & 3 \arrow[ur] & & & \\ & 2 \arrow[r] & 5 & & \end{tikzcd}$$

From this diagram it is clear the only way to satisfy the problem is to start with $\underline{A} = 8$, and so the answer is that there is one such number and it is 81649.

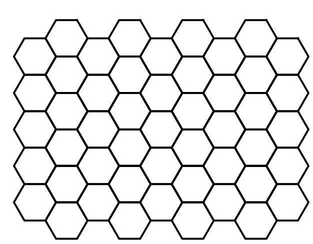
2. In the triangle $\triangle ABC$ we have $AB = 20$, $AC = 21$ and $BC = 29$. The points E and D are on the side $\overline{BC}$ so that $BD = 8$ and $EC = 9$. Compute the angle $\angle DAE$.

SOLUTION. First notice that $841 = 29^2 = 20^2 + 21^2 = 400 + 441$. This means that in ΔABC, $BC^2 = AB^2 + AC^2$, which means $\angle BAC = 90°$.

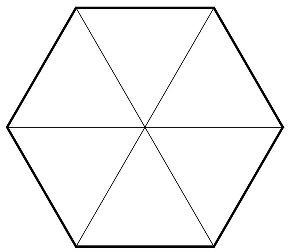


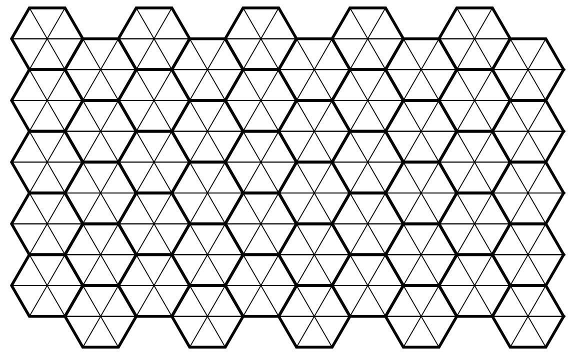
Next, we see that length of DE is equal to $29 - 8 - 9 = 12$, and thus $BE = 20$ and $CD = 21$. This means that that ΔBAE and ΔCAD are isosceles. Assume that $\angle ABC = x°$. Then $\angle ACB = (90 - x)°$. Since ΔBAE is isosceles, we have $\angle BAE = \left(90 - \frac{x}{2}\right)^{\circ}$. Similarly, since ΔCAD is isosceles, we have $\angle CAD = \left(45 + \frac{x}{2}\right)^{\circ}$. Finally, we have that $\angle BAE + \angle CAD = \angle BAC + \angle DAE$. Substituting the available angle values, we obtain $90 - \frac{x}{2} + 45 + \frac{x}{2} = 90 + \angle DAE$, which gives $\angle DAE = 45°$.

3. A plane is tiled with regular hexagons. This tiling is sometimes called the 'honeycomb' lattice. (The picture on the right shows a portion of the tiling.) Show that if a line passes through two points that are vertices of hexagons in the tiling, then the line passes through infinitely many such points.



SOLUTION. We may assume that the tessellation is aligned so that each regular hexagon has a horizontal edge (as it shown in the diagram above). We can divide each hexagon into six congruent equilateral triangles by connecting its center with its vertices. This way we get a tessellation of the place with equilateral triangles, this produces a so-called triangular lattice. The lines in the tessellation (which we call lattice lines) are either horizontal or have angles 60^o or 120^o with the horizontal.

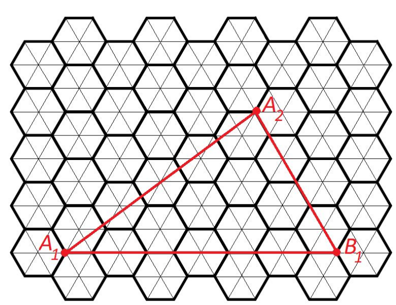




The triangular tessellation has two kind of vertices: points that were vertices of the hexagons (let's call these good vertices) and vertices that were centers of hexagons (let's call these bad vertices). By looking at a single hexagon we see that for each each bad vertex (hexagon center) there are six good vertices that are of distance one from it in both directions on all three lattice lines that go through it. Since each good vertex is the vertex of three hexagons, we see that for each good vertex there are one good and one bad vertices that are of distance one on each lattice line going through it. This shows that on any given lattice line we have 2 good vertices followed by a bad vertex, with this pattern repeating in both directions (and with the neighboring points being unit distance from each other).

Now suppose that the line ℓ passes through two distinct hexagon vertices, A_1 and A_2. If ℓ is one of the lattice lines then it will go through infinitely many good vertices by the argument above.

If ℓ is not one of the lattice lines then the horizontal lattice line going through A_1 and a non-horizontal lattice line going through A_2 will have an intersection point B_1, producing a triangle $A_1B_1A_2$. Since B_1 is the intersection of two lattice lines, it has to be a vertex of the triangular lattice, and $\angle A_1B_1A_2$ is either 60^o or 120^o.



Since A_1, B_1 are on a horizontal lattice line and they are both vertices, their distance is an integer. Similarly, A_2, B_1 are on the same lattice line and they are both vertices in the lattice, so their distance is an integer.

Let point A_3 lie on line ℓ so that A_2 is the midpoint of $\overline{A_1A_3}$. Let point B_2 lie on the horizontal line through A_2 so that $A_2B_2 = A_1B_1$ and $\angle A_3A_2B_2 = \angle A_2A_1B_1$. Then $\triangle A_2B_2A_3$ is congruent to $\triangle A_1B_1A_2$ by side-angle-side. But that means that $A_2B_2 = A_1B_1$ is an integer, hence B_2 (which is on the same horizontal lattice line as A_2) is a vertex in the triangular lattice. But $A_3B_2 = A_2B_1$ is also an integer, A_3 is on the same lattice line as B_2, hence A_3 is also a vertex in the lattice.

Similarly, construct a sequences of points $A_4, A_5, A_6, \ldots$ on ℓ, and points $B_3, B_4, B_5 \ldots$ so that for each n, $\triangle A_{n+1}B_{n+1}A_{n+2}$ is congruent to $\triangle A_nB_nA_{n+1}$, and both the A_n and B_n points are vertices of the triangular lattice.

This way we produced infinitely many points $A_1, A_2, A_3, \ldots$ on the line ℓ so they are all vertices. We just have to make sure that there are infinitely many good vertices among them. We claim that for any positive integer n the vertex A_{3n+1} must be 'good', which means that the points $A_1, A_4, A_7, \ldots$ are all vertices in the original hexagonal lattice.

Let C_{3n} be the intersection point of the horizontal line going through A_1 and the lattice line going through A_{3n+1} that is parallel to A_2B_1. Then the triangle $A_1C_{3n}A_{3n+1}$ is similar to the triangle $A_1B_1A_2$ since they have the same angles. By construction, A_1A_{3n+1} is $3n$ times A_1A_2, so this is true for the other sides as well: $A_1C_{3n} = 3n \cdot A_1B_1$ and $A_{3n+1}C_{3n} = 3n \cdot A_2B_1$. A_1B_1 is an integer, so A_1C_{3n} is an integer multiple of 3. Since A_1, C_{3n} are on a horizontal lattice line, and A_1 is a good vertex, C_{3n} must be a good vertex as well. (Because on each lattice line two good vertices are followed by a bad vertex, and this pattern repeats.) The same argument shows that $A_{3n+1}C_{3n}$ is also an integer multiple of three. Since C_{3n} is a good vertex, and A_{3n+1}, C_{3n} are on a lattice line we get that A_{3n+1} must be a good vertex as well. This shows that the points $A_1, A_4, A_7, \ldots$ are all vertices in the original hexagonal lattice, and hence ℓ goes through infinitely many vertices of the hexagonal lattice.

4. We know that a and b are positive numbers that satisfy $1/4 < a(1-b)$. Is it possible to determine from this information which of the numbers a or b is larger?

SOLUTION.

First, we claim that $b(1-b) \le 1/4$. This can be proven by expanding $(b-1/2)^2 \ge 0$ into $b^2 - b + 1/4 \ge 0$. Since $a(1-b) > 1/4$ and a is positive, it follows that $(1-b)$ is positive. Thus $a(1-b) > 1/4 \ge b(1-b)$ implies that $a > b$.

5. We call a positive integer 'interesting', if each one of its digits is either 0, 1 or 2, and any two of its neighboring digits differ by at most one. (For example, the number 1210012 is interesting, but the number 1200 is not.) Show that the number of n-digit interesting numbers is at most $\left(\frac{5}{2}\right)^n$.

SOLUTION. Let a_n denote the number of n-digit interesting integers. We first show that $a_{n+1} = 2a_n + a_{n-1}$ for all $n \ge 2$.

If we delete the last digit of an interesting n-digit number, then we get an interesting number of length $n-1$. Let x_n, y_n, z_n denote the number of n-digit interesting numbers that end with a digit of 0, 1, and 2, respectively. Then we have

$$a_n = x_n + y_n + z_n$$

for all $n \ge 1$.

If an interesting number of length $n+1$ ends in 0, then its nth digit is either 0 or 1, and we can get all these numbers (each exactly once) by adding an extra 0 digit at the end of a length n interesting number that ends in 0 or 1. This means that $x_{n+1} = x_n + y_n$. Similarly, we can get all interesting numbers of length $n+1$ ending with 2 by considering the length n interesting numbers ending in 1 and 2 and putting an extra 2 at the end. This gives $z_{n+1} = y_n + z_n$. Finally, the interesting numbers of length $n+1$ can be obtained by taking any interesting number of length n and putting a 1 in the end. This shows that $y_{n+1} = a_n$.

Hence for any $n \geq 2$ we have

$$\begin{aligned} a_{n+1} &= x_{n+1} + y_{n+1} + z_{n+1} = (x_n + y_n) + a_n + (y_n + z_n) \\ &= (x_n + y_n + z_n) + a_n + y_n = 2a_n + a_{n-1}, \end{aligned}$$

where in the last step we used $a_n = x_n + y_n + z_n$ and $y_n = a_{n-1}$, which both hold if $n \geq 2$.

To prove that $a_n \leq (5/2)^n$ we use induction. We have two single digit interesting numbers: 1 and 2, and five two-digit interesting numbers: 10, 11, 12, 21, 22. Hence $a_1 = 2$, and $a_2 = 5$. We have $a_1 \leq (5/2)$ and $a_2 \leq (5/2)^2 = 6 + \frac{1}{4}$. Now assume that we have proved the inequality $a_n \leq (5/2)^n$ for all n up to some integer $m \geq 2$. Then

$$a_{m+1} = 2a_m + a_{m-1} \leq 2 \cdot (5/2)^m + (5/2)^{m-1} = (5/2)^m \left(2 + \tfrac{2}{5}\right) = (5/2)^m \cdot 2.4 \leq (5/2)^{m+1},$$

where we used the induction assumption for m and $m - 1$. This shows the inequality for $m + 1$, from which it follows for all integers that are at least 2. This completes the proof of $a_n \leq (5/2)^n$ for all $n \geq 1$.