WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET III (2018-2019)

1. Do there exist distinct positive integers B, U, C, K and Y, that satisfy the equality on the right?

$$B + \frac{1}{U + \frac{1}{C + \frac{1}{K + \frac{1}{Y}}}} = \frac{22703}{7177}$$

SOLUTION. Since all the numbers are greater than or equal to 1, we have that

$$U + \frac{1}{C + \frac{1}{K + \frac{1}{Y}}} > 1,$$

and its reciprocal is strictly smaller than 1. Thus, B must equal the largest integer smaller than $\frac{22703}{7177}$, which leads to $B = 3$. Subtracting B from $\frac{22703}{7177}$ and taking the reciprocal of both sides, we arrive at the equality

$$U + \frac{1}{C + \frac{1}{K + \frac{1}{Y}}} = \frac{7177}{1172}.$$

Again, using the fact that all the unknown numbers are at least 1, we conclude that U must equal the largest integer smaller than $\frac{7177}{1172}$. This leads to $U = 6$. Subtracting U from both sides and taking the reciprocal yields

$$C + \frac{1}{K + \frac{1}{Y}} = \frac{1172}{145}.$$

The value of C must be the largest integer smaller than $\frac{1172}{145}$, giving $C = 8$. Subtracting C from both sides and taking the reciprocal gives

$$K + \frac{1}{Y} = \frac{145}{12}.$$

This gives $K = 12$ and $Y = 12$. Note that this solution is unique. Since $K = Y$, this means that there cannot exist distinct integers B, U, C, K and Y that satisfy the equality stated in the question.

2. Show that the numerical value of the expression $\dfrac{2 \cdot 4 \cdot 6 \cdots 2018 \cdot 2020}{1 \cdot 3 \cdot 5 \cdots 2017 \cdot 2019}$ is between 44 and 64.

SOLUTION. For any number k we have $k^2 > k^2 - 1 = (k-1)(k+1)$. Squaring the fraction and using the inequality for each factor in the numerator we get

$$\frac{2^2 \cdot 4^2 \cdot 6^2 \cdots 2018^2 \cdot 2020^2}{1^2 \cdot 3^2 \cdot 5^2 \cdots 2017^2 \cdot 2019^2} > \frac{(1 \cdot 3)(3 \cdot 5)(5 \cdot 7) \cdots (2017 \cdot 2019)(2019 \cdot 2021)}{1^2 \cdot 3^2 \cdot 5^2 \cdots 2017^2 \cdot 2019^2} = 2021,$$

where the last equality comes from canceling the squares of the odd integers from 3 to 2019. Since $2021 > 1936 = 44^2$, taking square roots shows that the original fraction is larger than 44.

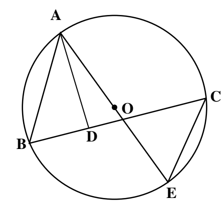
Using now $k^2 > k^2 - 1 = (k-1)(k+1)$ in the denominator, and then cancelling all the even numbers from the denominator we get

$$\frac{2^2 \cdot 4^2 \cdot 6^2 \cdots 2020^2}{1^2 \cdot 3^2 \cdot 5^2 \cdots 2019^2} < \frac{2^2 \cdot 4^2 \cdot 6^2 \cdots 2018^2 \cdot 2020^2}{1^2 \cdot (2\cdot 4) \cdot (4 \cdot 6) \cdots (2016 \cdot 2018) \cdot (2018 \cdot 2020)} = 2 \cdot 2020 = 4040 < 4096 = 64^2.$$

Taking the square root gives the required upper bound.

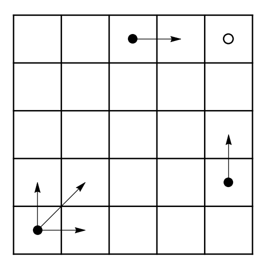
Remark. Using the estimate $k^2 > (k-1)(k+1)$ only for values k that are larger than a fixed value can give even better upper and lower bounds. For example, if we use it only for $k \geq 4$ in the first step, then we get the lower bound $\frac{2^2 \cdot 2021}{3} = 2694\frac{2}{3} > 51^2$ for the square of the expression, yielding a lower bound of 51 (instead of 44). The actual numerical value of the expression is approximately 56.3364.

3. The points A, B, C, E are on the boundary of a circle with center O. The point D is on the line segment BC, and AD is perpendicular to BC. We know that $AB = 12$, $BD = 6$ and $CE = 8$. Find the radius of the circle.



SOLUTION. The first step is noticing that ΔADB is a right-angled triangle with $\angle BAD = 30°$ and $\angle ABD = 60°$. This is because $BD = 6$ is half of the hypotenuse $AB = 12$. The second step is observing that ΔACE is also a $30° - 60° - 90°$ triangle. This is because $\angle ACE = 90°$ since it is the angle inscribed in a semi-circle, and $\angle ABC = \angle AEC = 60°$ since both these angle subtend the same arc. (See our page `https://goo.gl/pqq32m` for more information about various angles in the circle.) Therefore, the hypotenuse AE must be twice the length of CE, giving the length of $AE = 16$. The radius of the circle is half of the diameter AE, and must equal 8.

4. Ariana and Brooke play a game. They have a game board with an $n \times n$ grid of squares and a game piece in the bottom left square. The two players move the game piece one after another, beginning with Ariana. In each move, the game piece can be moved one step either up, right, or in the up-right diagonal direction, as long as it stays on the board. The first player to move the game piece into the top right corner square wins the game. For which $n > 1$ will Ariana have a winning strategy?



SOLUTION. We will show that if n is even, then Ariana has a winning strategy, and if n is odd, then Brooke has a winning strategy. We denote the squares of the board by cartesian coordinates so that the bottom left square is $(1,1)$. First, consider the case when n is even. We show, by induction, that Ariana has a strategy so that she always moves the piece to a square that has both coordinates even, and Brooke is never able to move the piece to such a square. Ariana's first move can be to $(2,2)$. Suppose Ariana has, on some move, moved the piece to $(2a, 2b)$. Then Brooke's responses are to move the piece to $(2a+1, 2b), (2a, 2b+1)$, or $(2a+1, 2b+1)$ (as long as these are on the board), none of these have booth coordinates even, and Ariana can then move to $(2a+2, 2b), (2a, 2b+2)$, or $(2a+2, 2b+2)$, respectively, all of which have both coordinates even. We also note that since the board is $n \times n$ and n is even, for each of Brooke's possible moves, if it is on the board, then so is Ariana's response. Thus, Ariana can follow this strategy, and the sum of the coordinates of the piece keeps increasing, and eventually the piece will thus have to be moved to (n, n), which can only be after Ariana's move, because of her strategy.

In the second case, if n is odd, we show that Brooke has a winning strategy by always moving the piece to a position with both coordinates odd, and never letting Ariana do so. On Ariana's first move, she must move to $(2,1), (1,2)$, or $(2,2)$, and then Brooke can move to $(3,1), (1,3)$, or $(3,3)$, respectively. Suppose Brooke has, on some move, moved the piece to $(2a+1, 2b+1)$. Then Ariana's responses are to move the piece to $(2a+2, 2b+1), (2a+1, 2b+2)$, or $(2a+2, 2b+2)$ (as long as these are on the board), none of these have booth coordinates odd, and Brooke can then move to $(2a+3, 2b+1), (2a+1, 2b+3)$, or $(2a+3, 2b+3)$, respectively, all of which have both coordinates even (and are on the board when Ariana's move was on the board since n is odd). Thus, similarly to the case above, the piece eventually moves to (n, n), which will have to have been after Brooke's move, and Brooke has a winning strategy.

5. A cube has one red and five white faces. We place it on a table so that it rests on the red face. An "edge move" is a $90°$ rotation of the cube around one of the four edges resting on the table, so that a new face will then be resting on the table. Suppose somebody performs 12 edge moves, so that each one of the cube's 12 edges is used exactly once as an axis of rotation. Prove that after the final move, the cube will again rest on the red face.

SOLUTION. Let's call the face of the cube that is on the table the 'bottom face'. We call two faces neighbors if they are joined by an edge. When we make an edge move the bottom face changes: it moves to one of its four neighboring faces, and the axes of rotation in the rotation is the edge connecting the two faces.

Consider the four edges on the red face of the cube. We have to use one of them as the axis of rotation in the first edge move, suppose that we used the other three at moves $a < b < c \leq 12$. After the first move the bottom face changes from red to white. It cannot change back to red unless we use one of the edges on the red face, which happens at move a. After this move the bottom face will change back to red. (Because the bottom is white before that edge is used, and we use an edge which connects this white face to the red one.) In the next move (move $a+1$) we have to change the bottom face from red to white again, so we have to use one of the unused edges on the red face. This also means that $b = a + 1$. Right after this move, the bottom becomes white, and it will stay white until we use the last of the four edges of the red face (in move c). Thus after move c the bottom becomes red again. But now we used up all four of the edges on the red face, so we cannot make another edge move (since each edge can only be used once). Hence, this was our last move and $c = 12$. But that shows that after the last move, the bottom face is red again.

Here is schematic representation of how the color of the bottom face changes (R=red, W=white)

$$R \xrightarrow{1} W \to \cdots \to W \xrightarrow{a} R \xrightarrow{a+1} W \to \cdots \to W \xrightarrow{12} R$$

The edge moves corresponding to the four edges of the red face happen at moves 1, $a, a+1$ and 12. (Here $2 < a < 10$.)