WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET II (2018-2019)

1. There is a book club with six members. They each bring a different book to the club meeting, and each person loans their book to another person in the club so that each person leaves with one borrowed book. At the last meeting the book borrowed by Angelica belonged to the borrower of Ben's book. The owner of the book borrowed by Cathy borrowed the book belonging to the borrower of Derek's book. The borrower of Elena's book was not the owner of the book borrowed by Fatima. From these clues, can we determine who borrowed Angelica's book? Can we determine whose book Angelica borrowed?

SOLUTION. We will write $X \rightsquigarrow Y$ if Y borrowed X's book, and we will abbreviate people's names by their first initial, and use $x, y, \ldots$ for unknown people. The first clue tells us $B \rightsquigarrow x \rightsquigarrow A$, and we have $x \neq A, B$ since each person borrows a book. The second clue tells us $D \rightsquigarrow y \rightsquigarrow z \rightsquigarrow C$. We know $D \neq y$, $y \neq z$ and $z \neq C$ since each person borrows a book (but so far it is possible that $y = C$ and $z = D$). The final clue tells use that $E \rightsquigarrow w$ and $v \rightsquigarrow F$, where $w \neq v$.

We will organize the people into circles such that each person is to the right of the person they borrowed a book from. We will explain more concretely how to do this in an example. Consider the chain of people that starts with Derek, then follows his book to whoever borrowed it and that person is added to the list, then follows that person's book to who borrowed it and adds them to the list, etc. In other words, we follow the arrows $\rightsquigarrow$ starting from D. Since there are only finitely many people, eventually there has to be a repeated name on the list. We claim that Derek will be the first repeated name. Let u denote the first repeated person, and assume that $u \neq D$. Then we have a chain $D \rightsquigarrow \cdots \rightsquigarrow u \cdots \rightsquigarrow u$. Now consider the person who is lending the book to u, this person must be exactly in front of u in the chain. Since u is not the first in the chain (as $u \neq D$), the person lending the book to u must appear twice, hence u cannot be the first repeated name. Thus, Derek is the first repeated name, and, thus, our chain closes into a circle. So we have a circle of people starting with Derek and coming back around to Derek on this list, and we know from the above clues that Cathy is on this list.

First we consider the case that this circle has only two people D, C. Then similarly to the circle starting with D, we have a circle starting with B. That circle has at least three people, B, x, A. If that circle has exactly 3 people, there would be one person left who would not have loaned or borrowed a book, which is a contradiction. So the circle with B and A must contain 4 people, meaning that we have $B \rightsquigarrow x \rightsquigarrow A \rightsquigarrow q \rightsquigarrow B$ with a q that is different from A, B, x, C, D. Then we must have $x = E, q = F$ or $x = F, q = E$, and both of these cases lead to a contradiction with the assumption that $E \rightsquigarrow w$, $v \rightsquigarrow F$ with $w \neq v$.

Thus, the list of people created above starting with D, has at least three people, and so $y \neq D, C$. Also, $z \neq D, C$, and, thus, we see that the circle starting with Derek has at least 4 people. However, that means that B and A must be in this circle (since B's circle has at least 3 people and there are 6 people total). If the circle containing D has only 4 members then it must be $D \rightsquigarrow y \rightsquigarrow z \rightsquigarrow C \rightsquigarrow D$, and then we would not be able to fit both A and B (which have a third person between them). Thus, the circle containing D has at least 5 people. However, if it had 5, then one person would not have borrowed a book from anyone, and we conclude the circle containing D has all 6 people. Thus, the circle is $D \rightsquigarrow y \rightsquigarrow z \rightsquigarrow C \rightsquigarrow s \rightsquigarrow t \rightsquigarrow D$, where y, z, s, t are equal to A, B, E, F in some order. Since we have $B \rightsquigarrow z \rightsquigarrow A$, that means either $A = y$, $B = t$ or $A = s$, $B = z$ since these are the only ways to choose two unassigned persons with exactly one person between them. If $A = y$, $B = t$, then we cannot have $E = z$, since that would mean $F = s$

and $E \rightsquigarrow C \rightsquigarrow F$ (violating the third clue). So in this case $A = y$, $B = t$, $E = s$ and $F = z$ and we have $D \rightsquigarrow A \rightsquigarrow F \rightsquigarrow C \rightsquigarrow E \rightsquigarrow B \rightsquigarrow D$.

If $A = s$, $B = z$ then similarly to the last case, we cannot have $E = t$, so $E = y$ and $F = t$, and we get $D \rightsquigarrow E \rightsquigarrow B \rightsquigarrow C \rightsquigarrow A \rightsquigarrow F \rightsquigarrow D$. Both of the found possibilities satisfy all the clues. In both cases Fatima borrowed Angelica's book, but Angelica borrows from different persons in the two cases (Derek and Cathy).

Thus, we can determine who borrowed Angelica's book, but not whose book Angelica borrowed.

2. What is the smallest possible three digit positive integer k which has a six digit integer multiple of the form $ABABAB$? (Here A and B are decimal digits, not necessarily distinct.)

SOLUTION. The number $ABABAB$ can be written as

$$\begin{aligned} ABABAB &= A\cdot 100000 + B\cdot 10000 + A\cdot 1000 + B\cdot 100 + A\cdot 10 + B \\ &= A\cdot 101010 + B\cdot 10101 = (10A + B)\cdot 10101. \end{aligned}$$

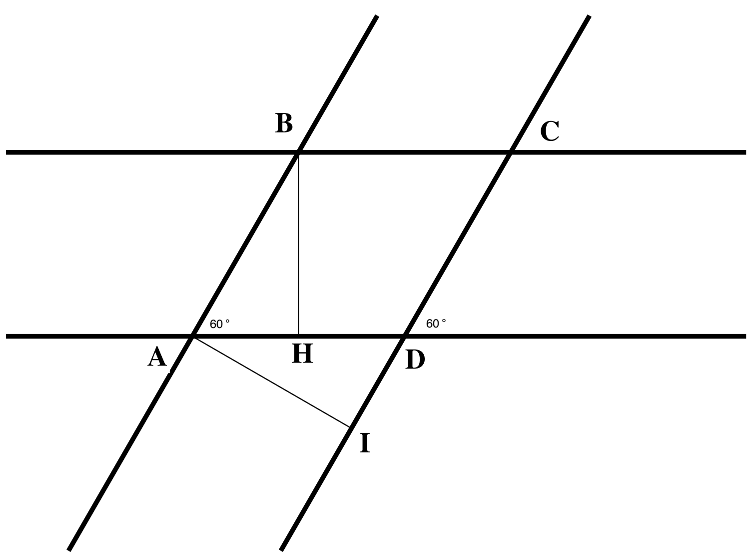
Since k has three digits, we have $k \geq 100$. If $k = 100$, then all of its integer multiples would end in two zeros. Such a number cannot be of the form $ABABAB$, because then we had $A = B = 0$.

This means $k \geq 101$. Note that 101 is a prime number. Thus, if 101 has an integer multiple that is equal to $ABABAB = (10A+B)\cdot 10101$, then either $10A+B$ or 10101 would be divisible by 101. $10A+B$ is a two digit integer, hence it is not a multiple of 101. We also have $10101 = 10000+101$ and since 10000 is not a multiple of 101, neither is 10101. Hence k cannot be equal to 101, which means that $k \geq 102$.

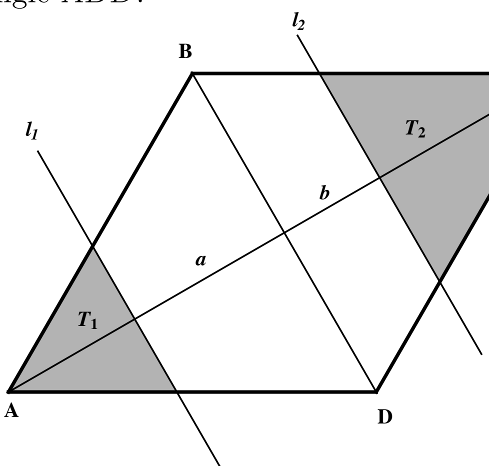
The number 102 has an integer multiple of the desired form: $343434 = 3367\cdot 102$. This means that $k = 102$ the smallest integer satisfying the conditions of the problem.

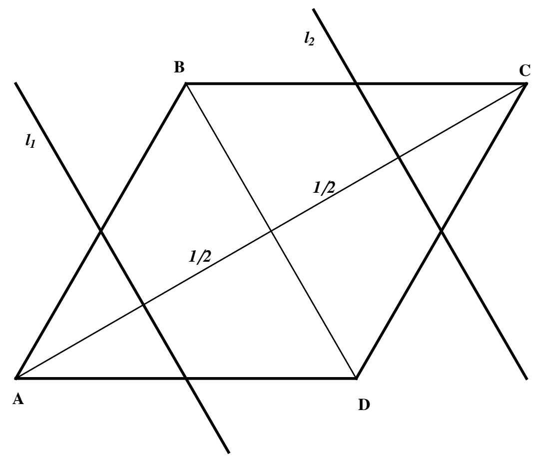
3. Suppose we have three extremely long strips of paper, each with width 1 cm and each having parallel sides. They are placed on a large table so that they are making angles of 0°, 60°, and 120° with the horizontal. Consider the region that is covered by all three strips. What is the largest possible area of this region?

SOLUTION. Consider the two strips with angles 0° and 60°. The intersection of these strips form a parallelogram, so denote its vertices by A, B, C, D, as shown below.



Let H be the foot of the perpendicular from B to line AD. Then $BH = 1$, and since ABH is a $30° - 60° - 90°$ triangle, we have $AB = 2/\sqrt{3}$. Similarly, if I is the foot of the perpendicular from A to CD, we have $AI = 1$ and $AD = 2/\sqrt{3}$. Thus, $ABCD$ is a rhombus. Since $AB = AD$, we have that ABD is an isosceles triangle with base BD, and, thus, since $\angle BAD = 60°$, we have that $\angle ABD = \angle ADB = 60°$. Thus, ABD is equilateral, and $BD = 2/\sqrt{3}$. We have that AC is the perpendicular bisector of BD since $ABCD$ is a rhombus, and $AC = 2$, since it is twice the height of triangle ABD.





The third strip thus has edges parallel to BD. Suppose its edges, lines ℓ_1 and ℓ_2 are distances a and b from BD, with ℓ_1 closer to A and ℓ_2 closer to C. We have $a + b = 1$ by the given width of the strip. If ℓ_1 or ℓ_2 intersected line AC outside of segment AC, then the area could be increased by moving the strip so that both of ℓ_1 and ℓ_2 intersect segment AC. Then the area covered by all three strips is the area of the parallelogram minus the areas of the triangles T_1 and T_2 form by lines AB, AD and ℓ_1, and CB, CD and ℓ_2, respectively. Since ℓ_1 is parallel to BD, we have that ℓ_1 also makes $60°$ angles with AB and AD, and, thus, T_1 is equilateral with height $1 - a$ (with base $2(1-a)/\sqrt{3}$) and area $(1-a)^2/\sqrt{3} = b^2/\sqrt{3}$. Similarly, T_2 has area $b^2/\sqrt{3}$. To maximize the area covered by all three strips, we must minimize the area in T_1 and T_2, i.e. minimize $a^2 + (1-a)^2 = 2a^2 - 2a + 1 = 2(a - 1/2)^2 + 1/2$. Since $(a - 1/2)^2 \geq 0$, this expression is smallest when $a = 1/2$.

The area of the triangle ABD (and BCD) is equal to $1/\sqrt{3}$ and the area of the rhombus $ABCD$ is $\frac{2}{\sqrt{3}}$. When $a = b = 1/2$, the area of the triangles that we cut off are both equal to $\frac{(\frac{1}{2})^2}{\sqrt{3}} = \frac{1}{4\sqrt{3}}$. Thus, the maximal covered area is $\frac{2}{\sqrt{3}} - \frac{2}{4\sqrt{3}} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$. (It can be checked that in that case the covered region is a regular hexagon.)

4. We have 2018 nonnegative numbers which are all at most 1. For each pair of these numbers we find the absolute value of their difference. Then we add up all the resulting differences. What is the largest possible value of this sum?

SOLUTION. Denote the 2018 numbers in order by $a_1 \leq a_2 \leq \cdots \leq a_{2018}$. For each $1 \leq i \leq 2018$ we have $0 \leq a_i \leq 1$, and for $1 \leq i < j \leq 2018$ we have $a_i \leq a_j$. Thus, we have to add the numbers $a_j - a_i$ for all pairs i, j with $1 \leq i < j \leq 2018$. For a given $1 \leq k \leq 2018$ the number a_k will appear in 2017 differences: in $k - 1$ of those it will appear with a + sign (these are the differences $a_k - a_1, a_k - a_2, \ldots a_k - a_{k-1}$), and in $2017 - (k-1) = 2018 - k$ it will appear with a − sign (these are the differences $a_{k+1} - a_k, a_{k+2} - a_k, \ldots, a_{2018} - a_k$). This means that in the sum of all differences the term a_k will show up with a multiplier $k - 1 - (2018 - k) = 2k - 2019$. Thus, we can write the sum of all differences as

$$(2 \cdot 1 - 2019) \cdot a_1 + (2 \cdot 2 - 2019) \cdot a_2 + \cdots (2 \cdot k - 2019) \cdot a_k + \cdots + (2 \cdot 2018 - 2019) \cdot a_{2018}. \qquad (*)$$

Note that the numbers $a_1, a_2, \ldots, a_{1009}$ will all have a negative multiplier in the sum (the numbers -2017, $-2015, \ldots, -1$) and since all a_i are non-negative, the contribution of these terms is at most zero:

$$-2017a_1 - 2015a_2 - \cdots - 3a_{1008} - a_{1009} \le 0.$$

The upper bound is achieved at $a_1 = a_2 = \cdots = a_{1009} = 0$. Similarly, the numbers a_{1010}, $a_{1011}, \ldots, a_{2018}$ all have a positive multiplier in the sum of differences (the numbers 1, 3, ..., 2017), and since the a's are all at most 1, the contribution of these terms can be bounded as

$$a_{1010} + 3a_{1011} + \cdots + 2017a_{2018} \le 1 + 3 + \cdots + 2017.$$

This upper bound is achieved if $a_{1010} = a_{1011} = \cdots = a_{2018} = 1$.

These bounds show that sum of all the differences is the largest if $a_1 = a_2 = \cdots = a_{1009} = 0$, $a_{1010} = a_{1011} = \cdots = a_{2018} = 1$, and the value of the sum in this case is equal to $1 + 3 + \cdots + 2017$. We can simplify this expression by noting that with the choices $a_1 = a_2 = \cdots = a_{1009} = 0$, $a_{1010} = a_{1011} = \cdots = a_{2018} = 1$ a difference $a_j - a_i$ will be nonzero if $i \le 1009$ and $1010 < j$, and in this case the difference is equal to 1. There are $1009 \cdot 1009$ such pairs $i < j$ (as there are 1009 choices for i, and 1009 choices for j), hence the maximal value of the sum of differences is $1009^2 = 1018081$.

5. We color the integers from 1 to 999 with red and blue, so that each integer gets one of these two colors. How many different colorings can we construct with the property that there are more red integers within the numbers $1, \ldots, 500$ than within the numbers $501, \ldots, 999$?

SOLUTION. If we have k integers then there are 2^k different ways we can color them with red and blue. This is because the first integer can get two possible colors, and for any specific coloring of the first ℓ integers the next one can be colored two different ways, doubling the number of colorings from ℓ to $\ell + 1$.

Because of this there are 2^{999} different colorings of the numbers 1, ..., 999 with red and blue if we have no restrictions. We will show that exactly half of these (2^{998}) have the property prescribed in the problem.

Let us call a coloring of $1, \ldots, 999$ 'nice' if it satisfies the condition described in the problem. We will show that changing the color of each number to its opposite (red$\rightsquigarrow$blue and blue$\rightsquigarrow$red) will change a 'nice' coloring into not nice, and a not nice coloring into nice. This will show that there are the same number of nice colorings as not nice colorings, hence the number of nice colorings is 2^{998}.

Consider a nice coloring, and denote the number of red integers within $1, \ldots, 500$ by x, and the number of red integers within $501, \ldots, 999$ by y. Since the coloring is nice, we have $x > y$. After switching all the colors the new coloring will have $500 - x$ red integers within $1, \ldots, 500$ and $499 - y$ red integers within $501, \ldots, 999$. But $x > y$ implies $499 - x < 499 - y$, and since these are integer numbers, we also get $500 - x \le 499 - y$. This shows that the switched coloring is not nice.

Similarly, if a given coloring is not nice, then with the same notation as before we have $x \le y$, which implies $499 - x \ge 499 - y$, and from this we get $500 - x > 499 - y$. Thus, in this case, if we switch all the colors, then the coloring becomes nice.

This proves that the number of nice colorings is exactly half of all colorings.