WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET I (2018-2019)

1. Find all solutions to the "crossnumber" puzzle using the given clues.

Across: (a) Sum of the digits in the number is 11.
(d) Square of a prime number.
(e) Consecutive digits in descending order.

Down: (a) Fourth power of a number.
(b) Square of a number.
(c) Product of the digits in the number is 216.
(f) Multiple of 11 that ends in 5.

a		b	c
		d	
e	f		

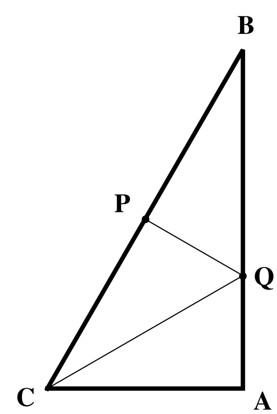
SOLUTION. The only two-digit multiple of 11 ending in 5 is 55, this gives (f) down. Now (e) across has to be 6 5 4 3. We have that (a) down is a fourth power that ends in 6. Since $6^4 = 1296 > 1000$ and $3^4 = 81 < 100$, we only need to check the fourth powers of 4 and 5 to see which one has its last digit equal to 6. The only solution is $4^4 = 256$, giving us (a) down. Now only two-digit squares of prime numbers are $5^2 = 25$ and $7^2 = 49$, so these are the choices for (d) across. However if (d) across is 25, then the product of the digits in (c) down will be divisible by 5, and 216 is not divisible by 5. This means that (d) across must be $7^2 = 49$. Using the hint for (c) down, we conclude that (c) down must be 8 9 3, since $8 \times 9 \times 3 = 216$. For (b) down, we need to find a three digit square number that ends in 44. Trying all 9 possibilities, we conclude that $12^2 = 144$ is the only possible solution. Finally, for (a) across, we use the hint that the sum of the digits in (a) across is 11 to conclude that (a) across must be 2 0 1 8.

a 2	0	b 1	c 8
5		d 4	9
e 6	f 5	4	3
	5		

2. Triangle ABC has angles $\angle ABC = 90^o$ and $\angle BCA = 60^o$. Show that $\triangle ABC$ can be divided into three congruent triangles.

SOLUTION. The ABC triangle has angles $30^o, 60^o$ and 90^o.

Let P be the midpoint of the side $\overline{BC}$. Consider the angle bisector of $\angle BCA$, and let Q be the intersection of the bisector with the side $\overline{AB}$. Since $\overline{CQ}$ is the bisector, we have $\angle QCA = \angle QCB = \frac{60^o}{2} = 30^o$. Thus, $\triangle ACQ$ has angles $30^o, 60^o$ and 90^o. Moreover, $\triangle QCB$ has angles 30^o at $\angle QCA$ and $\angle QBC$, which means that $\triangle QCB$ is an isosceles triangle and the third angle is $\angle CQB = 120^o$. Since $\triangle QCB$ is an isosceles, the line connecting Q with the midpoint P of $\overline{BC}$ is perpendicular to $\overline{BC}$: we have $\angle QPC = \angle QPB = 90^o$. But that means that the $\triangle QPC$ and $\triangle PB$ both have angles $30^o, 60^o$ and 90^o. This shows that the lines QC and PQ divide $\triangle ABC$ into three similar triangles $\triangle ACQ$, $\triangle QPC$, $\triangle QPB$ (since each one has the same three angles).



The fact that these triangles are congruent follows from the application of angle-side-angle comparisons. Indeed, $\triangle ACQ$ and $\triangle QPC$ share the side $\overline{QC}$ and the corresponding angles are equal: $\angle QCA = \angle QCP = 30^o$ and $\angle CQA = \angle CQP = 60^o$, hence these two triangles are congruent. Similarly, the $\triangle QPC$ and $\triangle QPB$ are congruent because they share the side $\overline{QP}$ and the corresponding angles are equal: $\angle QPC = \angle QPB = 90^o$ and $\angle PQC = \angle PQB = 60^o$.

3. Suppose that $r > 2$ is an integer. Show that $\sqrt{r} - \sqrt{2}$ cannot be an integer.

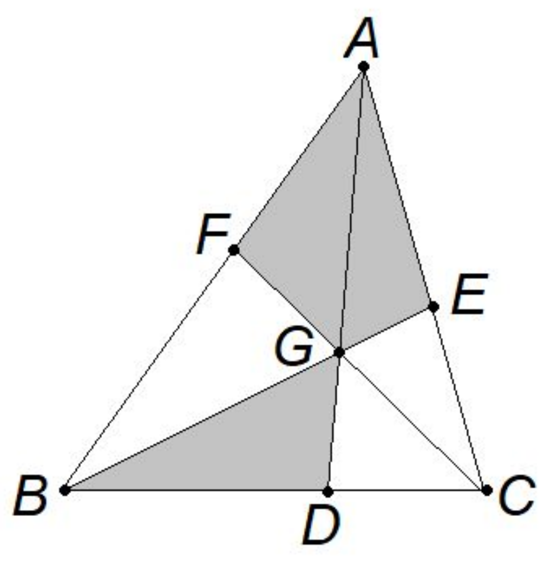
SOLUTION. Assume that $r > 2$ and $\sqrt{r} - \sqrt{2}$ are both integers. Then

$$\sqrt{r} + \sqrt{2} = \frac{(\sqrt{r} + \sqrt{2})(\sqrt{r} - \sqrt{2})}{\sqrt{r} - \sqrt{2}} = \frac{r-2}{\sqrt{r} - \sqrt{2}}$$

is the ratio of two integers, so it is a rational number. But then $(\sqrt{r} + \sqrt{2}) - (\sqrt{r} - \sqrt{2}) = 2\sqrt{2}$ is the difference of a rational number and an integer, so it is also a rational number implying that $\sqrt{2}$ is a rational number. This is a contradiction because $\sqrt{2}$ is irrational, so r and $\sqrt{r} - \sqrt{2}$ cannot both be integers. (In fact, they cannot even both be rational.)

4. Let G be a point in the interior of the triangle ABC, and let D, E, and F be points on sides $\overline{BC}$, $\overline{AC}$, $\overline{AB}$, respectively, such that $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ all intersect at G. Given that $\triangle AFG$, $\triangle AGE$, and $\triangle BDG$ all have the same area, show that D, E and F are the midpoints of the sides $\overline{BC}, \overline{CA}$, and $\overline{AB}$, respectively.

SOLUTION. Without loss of generality, let $\triangle AFG$, $\triangle AGE$, and $\triangle BDG$ all have area 1, and let $\triangle CEG$, $\triangle CGD$, and $\triangle BGF$ have areas x, y, and z, respectively. Note that $\triangle CGD$ and $\triangle BDG$ have the same altitude corresponding to G. If we denote the length of this altitude by h, then we have



$$y = \text{area}(\triangle CGD) = \frac{CD \cdot h}{2}, \quad 1 = \text{area}(\triangle BDG) = \frac{BD \cdot h}{2}$$

which means

$$\frac{y}{1} = \frac{\text{area}(\triangle CGD)}{\text{area}(\triangle BDG)} = \frac{CD}{BD}.$$

Similarly, the triangles $\triangle ADC$ and $\triangle ABD$ have the same altitude corresponding to A. Comparing the areas of these two triangles the same way as before we get

$$\frac{x+y+1}{1+1+z} = \frac{\text{area}(\triangle ADC)}{\text{area}(\triangle ABD)} = \frac{CD}{BD}.$$

Thus, we have

$$y = \frac{x+y+1}{2+z}.$$

Now comparing $\triangle CGE$ and $\triangle EGC$ (which have the same altitude corresponding to G), and then comparing $\triangle CBE$ and $\triangle EBA$ (which have the same altitude corresponding to B) we get

$$\frac{CE}{AE} = \frac{x}{1} = \frac{1+x+y}{2+z}.$$

But we also had $y = \frac{x+y+1}{2+z}$, so it must be that $x = y$.

Similarly as before, considering $\triangle BGF$, $\triangle GFA$, $\triangle CBF$ and $\triangle CFA$ we get

$$\frac{BF}{AF} = \frac{z}{1} = \frac{1+y+z}{2+x} = \frac{1+x+z}{2+x}.$$

Rearranging the equation we get $z(2+x) = 1+x+z$ which leads to

$$0 = zx + z - x - 1 = (z-1)(x+1).$$

Because $x > 0$, this means that z must be equal to 1. Going back to the equation $y = \frac{x+y+1}{2+z}$ we get $x = \frac{1+2x}{3}$ which leads to $x = 1$ and $y = 1$.

It then follows that $\frac{CD}{BD} = \frac{CE}{AE} = \frac{BF}{AF} = 1$, so the points D, E and F are the midpoints of the respective sides.

5. The numbers 1,2,. . . , 2018 are written in order in a long line on a big board. Somebody plays the following game: she chooses two numbers next to each other, erases them and writes the absolute value of the difference of the two numbers in place of them (only once). She repeats this until only one number is left on the board. Find all possible numbers that can be equal to this final number.

SOLUTION. We will show that the odd numbers $1, 3, 5, \ldots, 2017$ can be obtained at the end of the game, and nothing else.

We can only get nonnegative integer numbers on board using the steps in the game. Moreover, two absolute value of the difference of two nonnegative integers cannot be larger than any of the two numbers, which means that we can never get a number larger than 2018. Finally, note that the sum of all the numbers on the board does not change in parity. Indeed, she replaces two evens or two odds with an even, or an even and an odd with and odd. Since $1 + 2 + \cdots + 2018 = \frac{(2018)(2019)}{2} = 1009 \cdot 2019$ is odd, she cannot end with an even number. This means that the final number is nonnegative, at most 2018, and must be odd: it must be one of $1, 3, 5, \ldots, 2017$.

We now show how to obtain an odd number $1 \leq k \leq 2017$. If $k > 1$, we start by pairing the numbers strictly between 1 and $k + 1$, i.e. $(2, 3), (4, 5), \ldots, (k - 1, k)$, and replace each of the $(k - 1)/2$ pairs with difference 1. Then, if $k < 2017$, we pair the numbers after $k + 1$, i.e. $(k + 2, k + 3), \ldots, (2017, 2018)$, and replace each of the $(2017 - k)/2$ pairs with difference 1. So we have $(k + 1)/2$ copies of the number 1, followed by the number $k + 1$, and then followed by $(2017-k)/2$ copies of the number 1. Note that $(k+1)/2+(2017-k)/2 = 1009$, so one of $(k+1)/2$ and $(2017-k)/2$ is odd and the other is even. We take the even length string of 1's and pair them to make all 0's. (If $k = 2017$, then we don't do anything in this step.) In the odd length string of 1's, we pair the 1 that is adjacent to $k + 1$ with the $k + 1$, and replace the pair with k. Then we take the remaining even length string of 1's and pair them to make all 0's. Finally, we keep pairing k with a neighboring 0, which has the effect of deleting that 0, until all of the 0's are gone, and we are left with the number k on the board.