WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET V (2017-2018)

1. Find the largest perfect square that has no even digits.

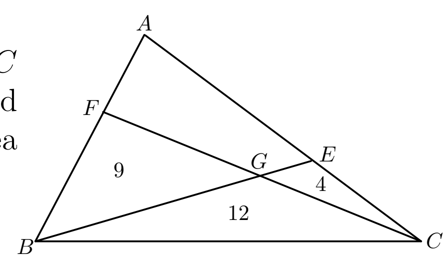
SOLUTION. The largest such number is 9. To prove this, we will show that if a perfect square has at least two digits then at least one of its last two digits will be even.

Suppose that $n \geq 4$, and consider n^2. If n is even then n^2 is also even and hence its last digit is even. We will show that if n is odd then the next to last digit of n^2 (the one at the tens place) is even. Note that if $n \geq 4$ then $n^2 \geq 16$ and there must be a digit at the tens place of n^2.

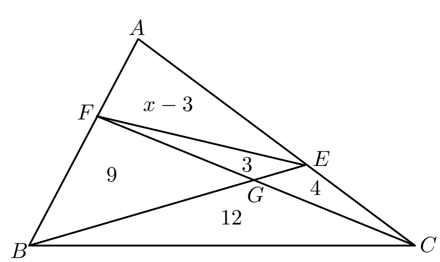
Let us write n as $n = 10a + b$ where b is a single digit odd number (i.e. 1, 3, 5, 7 or 9) and a is a nonnegative integer. (If n only has one digit then $a = 0$.) Then $n^2 = (10a+b)^2 = 100a^2+20ab+b^2$. The last two digits of $100a^2$ are 00, thus n^2 will have the same last two digits as $20ab + b^2$. b^2 is either $1, 9, 25, 49$ or 81, and in each of these numbers are either single digit numbers or have and even digit at the tens place. The number $20ab = 10 \cdot (2ab)$ is either zero (if $a = 0$), or its last digit is zero, and it's next to last digit is the same as $2ab$'s last digit, which is even. This means that when we add $20ab$ to b^2 then at the ones place we have to add 0 to an odd digit (which gives an odd answer, with no carries), and then in the tens place we have to add two even digits, which will result in an even digit at the tens place of n^2. Thus if n is odd and $n \geq 4$ then n^2 must have an even digit at the tens place.

This shows that if n^2 only has odd digits then $n \leq 3$, which implies that $3^2 = 9$ is the largest perfect square with no even digits.

2. In the adjoining figure, points E and F are chosen on the sides AC and AB of $\triangle ABC$ and G is the intersection of the segments BE and CF. The area of $\triangle BGF$ is 9, the area of $\triangle BGC$ is 12, and the area of $\triangle CGE$ is 4. Find the area of quadrilateral $AFGE$.



SOLUTION. Denote the area of quadrilateral $AFGE$ by x. Construct the segment EF as shown. Notice that $\frac{\text{area}(\Delta BFG)}{\text{area}(\Delta BGC)} = \frac{\text{area}(\Delta EFG)}{\text{area}(\Delta EGC)} \left(= \frac{FG}{GC}\right)$. This gives $\frac{9}{12} = \frac{\text{area}(\Delta EFG)}{4}$ and we conclude



that area$(\Delta EFG) = 3$. This also implies area$(\Delta AFE) = 3 - x$. To find x, we observe that $\frac{\text{area}(\Delta AFE)}{\text{area}(\Delta FBE)} = \frac{\text{area}(\Delta AFC)}{\text{area}(\Delta FBC)} \left(= \frac{AF}{FB}\right)$. This gives the equation $\frac{x+4}{21} = \frac{x-3}{12}$, and we conclude $x = \frac{37}{3}$

3. Show that

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2^{2018}-1}{2^{2018}} < \frac{1}{10^{100}}.$$

SOLUTION. Let

$$x = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2^{2018}-1}{2^{2018}} \qquad \text{and} \qquad y = \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{2^{2018}}{2^{2018}+1}.$$

Then

$$xy = \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}\cdot\frac{5}{6}\cdots\frac{2^{2018}-1}{2^{2018}}\cdots\frac{2^{2018}}{2^{2018}+1}.$$

Canceling a 2 from the numerator and denominator of xy, and then a 3, and then a 4, up through canceling a 2^{2018} from the numerator and denominator of xy, we see that $xy = 1/(2^{2018}+1)$. Now note that since $1/2 < 2/3$ and $3/4 < 4/5$, and in general

$$\frac{k}{k+1} = 1 - \frac{1}{k+1} < 1 - \frac{1}{k+2} = \frac{k+1}{k+2},$$

we have that $x < y$ by comparing their products term by term. So $x^2 < xy = 1/(2^{2018}+1) < 1/2^{2018}$, and $x < 2^{-1009}$. Since $2^{10} > 10^3$, we conclude $x < 10^{-300} < 10^{-100}$.

4. In an after-school checkers club, each student in the club plays each other student exactly once in a game of checkers (and each game ends with one student winning). Prove that after all the games, the students can line up in some way so that each student won her game of checkers against the student ahead of her in line (except the first student, who has no one ahead of her).

SOLUTION. We prove this by induction on the number of students. If there are two students, then the winner of the game can be second in line and the conditions of the problem are satisfied. Suppose that whenever there are n students, they can always line up so that everyone won against the person ahead of her in line. Now consider the case where there are $n+1$ students. If one student (Lauren) waits to the side while n of the them line up, by the inductive hypothesis we know that the remaining n students can line up so that everyone won against the person ahead of her in line. Now Lauren can start at the back of the line and find the first person from the back that she beat in checkers (Carla). Lauren should insert herself directly behind Carla. Lauren beat Carla in checkers, but since Carla was the first person from the back that Lauren beat, the person previously behind Carla (and now behind Lauren) beat Lauren. If Lauren did not win any of her games, she can stand in front of the line, and the person behind Lauren will have beaten her. We have seen that the new line satisfies the requirement, which completes the induction.

5. A group of students from a class go to a carnival. The students go one at a time, with each student arriving and leaving at different times. Once a student leaves, they don't come back. We know that for any three students in the class, at least two of them were at the carnival at the same time. Show that we can choose two time instances so that every student from the class is present at at least one of those times (and maybe both). A student is counted as "present" at the instant of their arrival or departure.

SOLUTION. Let n be the number of students in the class. Let the arrival times ordered in ascending order be $a^{(1)} < a^{(2)} < \cdots < a^{(n)}$, and let the departure times ordered in ascending order be $d^{(1)} < d^{(2)} < \cdots < d^{(n)}$. Consider the following two time instants: $t = a^{(n)}$ and $s = d^{(1)}$. Thus, t is the last time a student arrives at the carnival, and s is the first time a student leaves the carnival. We consider two cases:

Case 1: If $t \leq s$, consider the time t. At this time, it must be that every single student is at the carnival, since no one has left (it is before s) and everyone has arrived (last arrival is at t).

Case 2: Now consider the case when $s < t$. We claim that every student must be present either at time t or at time s. Let U be the student that arrives at time t, let V be the student that leaves at time s, and let W be any other student. Clearly, W arrives before time t and leaves after time s. If W arrives before or at s, then W must leave after s – which implies W is present at time s. Other the other hand, if W arrives between s and t, and also leaves between s and t, then the three students U, V, and W will be such that no two of them are present at the carnival at the same time. This contradicts the given condition, and this means that if W arrives between s and t then W must leave after or at t – which means W is at the carnival at time t.