WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET IV (2017-2018)

1. Find the number of pairs of positive integers m and n such that $m + n = 2018$, and the addition of m and n involves no carry. (For example, in the case of $1012 + 1006$ there is no carry in the addition, but in the case of $1009 + 1009$ there is a carry from the ones place to the tens place.)

SOLUTION. If $m+n = 2018$ results in no carry, then m has 9 possible ones digits (from 0 to 8) 2 possible tens digits (0 or 1), and 3 possible thousands digits. Two of these choices for m, 0 and 2018, make either m or n equal to 0. Thus, there are $9 \cdot 2 \cdot 3 - 2 = 52$ possible sums involving no carry.

2. For which values of $n \geq 1$ is the expression $2018^n + 2019^n + 2020^n + 2021^n + 2022^n$ divisible by 5?

SOLUTION.

A number is divisible by 5 if and only if its last digit is 0 or 5. The last digit of $2018^n + 2019^n + 2020^n + 2021^n + 2022^n$ can be determined by adding up the last digits of each of the summands, and taking the last digit of the resulting sum. The last digit of 2018^1 is 8. The last digit of 2018^2 can be obtained by taking the last digit of 2018×8, which is 4. Similarly, the last digit of 2018^3 is obtained by taking the last digit of 2018^2 times 8, which is 2. Continuing, we notice a recurring pattern for the last digit of 2018^n: $8 \to 4 \to 2 \to 6 \to 8 \to 4 \to 2 \to 6 \to \ldots$. Doing the same for the other numbers, we get the following patterns for the last digit:

$$\begin{aligned}
2018 &: \quad 8 \to 4 \to 2 \to 6 \to 8 \to 4 \to 2 \to 6 \to \ldots \\
2019 &: \quad 9 \to 1 \to 9 \to 1 \to 9 \to 1 \to 9 \to 1 \to \ldots \\
2020 &: \quad 0 \to 0 \to 0 \to 0 \to 0 \to 0 \to 0 \to 0 \to \ldots \\
2021 &: \quad 1 \to 1 \to 1 \to 1 \to 1 \to 1 \to 1 \to 1 \to \ldots \\
2022 &: \quad 2 \to 4 \to 8 \to 6 \to 2 \to 4 \to 8 \to 6 \to \ldots
\end{aligned}$$

For $n = 1, 2, 3$ and 4, we can add up the last digits to notice that for $n = 1, 2,$ and 3, the numbers are divisible by 5. Since the pattern repeats after every 4 steps, we conclude that for any number n of the form $4k + 1$, $4k + 2$ or $4k + 3$ where $k \geq 0$, the resulting sum is divisible by 5.

3. Show that if n is a positive integer, then

$$\sqrt[n]{n+1} < 1 + \frac{\sqrt{2}}{\sqrt{n}}.$$

SOLUTION. If $n = 1$, then we have $\sqrt{1+1} < 1 + \sqrt{2}$ which is true. From now on we assume that $2 \leq n$. We will prove that the nth power of the left side of the inequality is less than the nth power of the right side, that is $n + 1 < \left(1 + \frac{\sqrt{2}}{\sqrt{n}}\right)^n$. This will imply the required inequality.

We will show that if $n \geq 2$ and $a > 0$, then

$$(1 + a)^n \geq 1 + na + \tfrac{n(n-1)}{2}a^2. \qquad (*)$$

Then with $a = \frac{\sqrt{2}}{\sqrt{n}}$ we get

$$\left(1+\frac{\sqrt{2}}{\sqrt{n}}\right)^n \geq 1 + n \cdot \frac{\sqrt{2}}{\sqrt{n}} + \frac{n(n-1)}{2}\left(\frac{\sqrt{2}}{\sqrt{n}}\right)^2 = 1 + \sqrt{2}\sqrt{n} + \frac{n(n-1)}{2} \cdot \frac{2}{n} = 1 + \sqrt{2n} + n - 1 = n + \sqrt{2n}.$$

Since $n \geq 2$, we have $n + \sqrt{2n} \geq n + \sqrt{2 \cdot 2} = n + 2 > n + 1$, which gives the needed inequality.

Back to the proof of inequality $(*)$. By definition

$$(1+a)^n = \underbrace{(1+a)\cdots(1+a)}_{n \text{ times}}.$$

Imagine what happens when we expand all parentheses in the product on the right! We will take one of the two terms (1 or a) from each factors, multiply them together and then add the results up for all choices. When we do this, all terms are positive. Thus if we only add a couple of terms in the sum, then the result will be at least as large as the original sum.

If we choose the term 1 from each of the n factors, then the corresponding product is 1. If we choose a from exactly one of the factors (and 1 from all the others), then the corresponding product is a. Note that we can do that n different ways (as there are n factors, and we can choose a from each one), thus the sum of these terms is $n \cdot a$.

Now let us look at the terms where we choose a from exactly two of the factors (and 1 from all the others). The product for such a choice is always a^2. How many ways can we choose the two a terms? Imagine that we pick these one by one. We have n choices for the first choice, and in each case we have $n-1$ for the second one, which gives $n(n-1)$ choices. But this way we double counted each configuration (e.g. if we pick the kth and ℓth factors with $k < \ell$ then this was counted as k, ℓ and ℓ, k), so the actual number of configurations is half of that: $\frac{n(n-1)}{2}$. The contributions of these terms is this $\frac{n(n-1)}{2} \cdot a^2$.

The sum of the terms that we considered so far gives $1 + na + \frac{n(n-1)}{2}a^2$. Since there might be other terms (e.g. with three a terms if $n \geq 3$), the sum we found is at most as large as the total sum, $(1+a)^n$, which proves the inequality $(*)$.

4. Use each of the digits $4, 5, 6, 7, 8, 9$ exactly once to form two three digit numbers having the largest possible product. (Don't forget to give a complete justification that you have found the largest possible product.)

SOLUTION. We write $\underline{abc}$ for the number composed of the digits a, b, c in that order.

Since there are finitely many ways to assign the 6 digits to two three digit numbers, there must be an assignment (or more) that produces the largest. Pick one of the assignments that produces the largest product.

Consider two three-digit numbers composed of the six given digits. If we take each number and reorder the digits in decreasing order, then each number increases, and so the product increases. Thus, the solution we are looking for is the product of numbers $\underline{abc}$ and $\underline{fgh}$ with $a > b > c$ and $f > g > h$. So either a or f is 9, and without loss of generality, we assume $a = 9$.

Next we show that f must be equal to 8. Assume that $f < 8$, then we must use the digit 8 in $\underline{abc} = \underline{9bc}$, which means $b = 8$. Then f, c, g, h are all at most 7, and

$$\underline{abc} \cdot \underline{fgh} \leq 987 \cdot 777 = 766899.$$

On the other hand, if we have $f = 8$, then since $b > c \geq 4, g > h \geq 4$ we have $b \geq 5, g \geq 5$ and

$$\underline{abc} \cdot \underline{fgh} \geq 954 \cdot 854 = 833796 > 766899.$$

Thus we must have $f = 8$ to have the maximal product.

The next largest unassigned digit is 7, so we must have either $b = 7$ or $g = 7$. We will show that if $b = 7$ (meaning that the two numbers are $\underline{97c}$ and $\underline{8gh}$), then switching $b = 7$ and g (to get $\underline{9gc}$ and $\underline{87h}$) will increase the product. For this we have to show that $\underline{9gc} \cdot \underline{87h} - \underline{97c} \cdot \underline{8gh} > 0$. Writing out the three digit numbers as sums we get

$$\underline{9gc} \cdot \underline{87h} - \underline{97c} \cdot \underline{8gh} = (900 + 10g + c)(870 + h) - (970 + c)(800 + 10g + h)$$

and expanding the expression on the right gives

$$\underline{9gc} \cdot \underline{87h} - \underline{97c} \cdot \underline{8gh} = 7000 + 70c + 10gh - 1000g - 10cg - 70h.$$

Since c, g, h are all between 4 and 6, the last expression is at least

$$\begin{aligned} &7000 + 70c + 10gh - 1000g - 10cg - 70h \\ &\qquad \geq 7000 + 70 \cdot 4 + 10 \cdot 4 \cdot 4 - 6 \cdot 1000 - 10 \cdot 6 \cdot 6 - 70 \cdot 6 = 660 > 0. \end{aligned}$$

Thus in this case switching b and g would indeed increase the product, and hence we must have $g = 7$ to have the maximal product. Thus $\underline{fgh} = \underline{87h}$.

The next unassigned digit is 6, and it must be h or b. If $b = 6$, then (choosing the lowest possible value for the unassigned digits):

$$\underline{abc} \cdot \underline{fgh} \geq 964 \cdot 874 = 842536.$$

If $h = 6$, then we must have $b = 5, c = 4$ (since $b > c$) and this would give

$$\underline{abc} \cdot \underline{fgh} = 954 \cdot 876 = 835704 < 842536.$$

Thus we must have $b = 6$. This means that we must have 965 and 874 or 964 and 875 for the two numbers. But $965 \cdot 874 < 964 \cdot 875$, which means that the largest product is obtained from the two numbers 964 and 875.

5. Hexagon $ABCDEF$ is inscribed in a circle radius R. Its side lengths are $AB = CD = EF = 4$ and $BC = DE = FA = 13$. Find R. For full credit, your proof should not include any trigonometry.

SOLUTION. Let O be the center of the circle. Then the triangles $\triangle OAB$, $\triangle OCD$, $\triangle OEF$ are congruent as we have $AB = CD = EF$ and all other sides are equal to the radius R. This means that the angles $\angle AOB$, $\angle COD$, $\angle EOF$ are all equal. The same argument shows that $\angle BOC = \angle DOE = \angle FOA$. Since the angles around O add up to $360°$, we must have $\angle AOB + \angle BOC = 120°$. This implies (among others) $\angle AOC = \angle BOF = 120°$. By the Inscribed Angle Theorem (see e.g. here: `https://www.math.wisc.edu/wiki/index.php/Problem_Solver%27s_Toolbox#Angles_in_the_circle`)we also have $\angle FAB = 120°$

Extend line AF through A to the point G so that $\angle AGB = 90°$. Because $\angle FAB = 120°$, it follows that $\angle BAG = 60°$, and $\triangle AGB$ is a 30-60-90° triangle. The sides of such a triangle are proportional to $1, \frac{\sqrt{3}}{2}$ and $\frac{1}{2}$, respectively. Since $AB = 4$, it follows that $AG = 2$ and $BG = 2\sqrt{3}$.

The right angled triangle $\triangle BGF$ has side lengths $BG = 2\sqrt{3}$ and $GF = GA + AF = 2 + 13 = 15$. Applying the Pythagorean Theorem shows that $BF^2 = BG^2 + FG^2 = (2\sqrt{3})^2 + (15)^2 = 237$ and $BF = \sqrt{237}$.

Since $\angle BOF = \angle FOD = \angle BOD = 120°$, we have $BF = FD = BD$ and the triangle $\triangle BFD$ is equilateral. Connecting O with the three vertices of $\triangle BFD$ we get $OB = OF = OD = R$.

Noting that $\triangle BOD$ can be divided into two congruent 30-60-90° triangles we see $\frac{\sqrt{3}}{2}R = \frac{BD}{2} = \frac{\sqrt{237}}{2}$ which leads to $R = \frac{\sqrt{237}}{\sqrt{3}} = \sqrt{79}$.