WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET II (2017-2018)

1. Consider the number $a = 20172017\dots2017$ where the digits 2,0,1,7 repeat one thousand times. In this number only four of the possible 10 digits appear. Show that the number a has an integer multiple that contains all ten digits in its decimal representation.

SOLUTION. We show that if n is a positive integer, then it has an integer multiple which contains all 10 digits in its decimal representation.

Suppose that n has k digits, then $10^{k-1} \leq n < 10^k$. We first show that for any integer $1 \leq j \leq 9$, there is a multiple of n with $k+1$ digits that starts with the digit j. The $k+1$ digit numbers starting with j are exactly the numbers between $j \cdot 10^k$ and $(j+1)10^k - 1$, which are exactly 10^k consecutive integers. Since $n < 10^k$, one of these 10^k consecutive integers must be divisible by n. For each $1 \leq j \leq 9$ let a_j be a multiple of n with $k+1$ digits and first digit j. Now set

$$m = a_1 + a_2 \cdot 10^{k+1} + a_3 \cdot 10^{2(k+1)} + \cdots + a_9 10^{9(k+1)}.$$

This is the integer we get by writing down the digits of a_9 then a_8, …then a_1 next to each other. This is because each a_j has exactly $k+1$ digits, and $a_j 10^{(j-1)(k+1)}$ is just a_j with an extra $(j-1)(k+1)$ zeros at the end. The integer m is a multiple of n, since $a_1, a_2, \dots, a_9$ are all divisible by n. Moreover, m has all the digits of $a_1, a_2, \dots, a_9$, which means that it will have all nine positive digits. Multiplying this number by 10 results in an integer multiple of n that must have all ten digits.

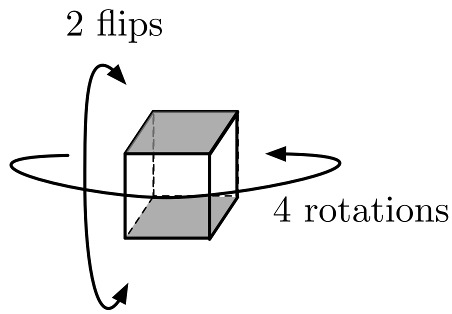
(The number given in the problem has 4000 digits, so the procedure above constructs a multiple with $9 \cdot 4001 + 1 = 36010$ digits that contains all ten possible digits.)

2. How many ways can you paint the six faces of a cube using the colors red, blue, yellow, green, pink, where each face is painted with a single color, and all 5 colors must be used at least once? Two colorings are considered the same if one may be obtained from the other by rotating the cube.

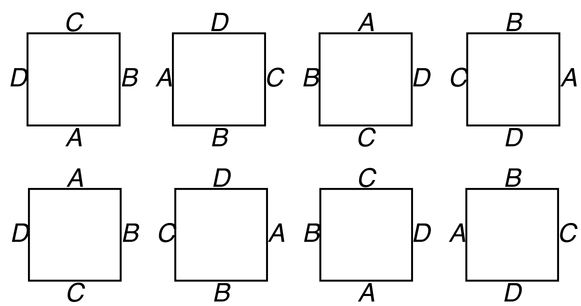
SOLUTION. Note that exactly one of the colors must be used twice, and all other colors are used once. The repeated color can be chosen 5 ways and for each specific color X there are the same number of colorings with X appearing twice. Thus it is enough to count the number of colorings with two red faces and we get the final answer by multiplying this number by 5.

If we have two red faces then these can either be opposite or adjacent faces of the cube. We will count these two cases separately.

In the first case we can construct a coloring the following way: we color two of the opposite faces of the cube red. Now imagine that you place the cube on the table so that it rests on one of the already colored red faces, and label the non-colored faces with A, B, C, D in order. We need to color the remaining 4 faces with the remaining 4 colors. We could do that by first choosing a color for side A (there are 4 choices for this), then choosing one of the three remaining colors for side B, then a color for side C and finally use the last unused color for side D. Collecting all the choices we made this way we get $4 \cdot 3 \cdot 2 \cdot 1 = 24$ different colorings. However, these colorings don't account for the colorings that can be rotated into each other!



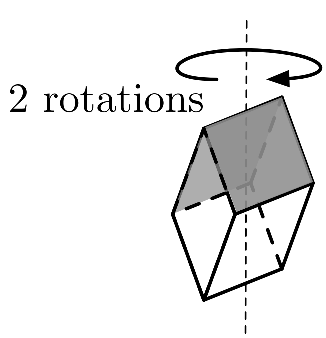
For any given coloring that we considered we can rotate the cube along its vertical symmetry axis by $90, 180, 270$ degrees and we would get 3 additional colorings that were counted separately (even though they are the same). Moreover, we may flip the cube upside down (that way we would still have red faces on the top and bottom) in each of these four colorings (the original plus the three new ones). These would produce another four colorings that we counted separately, even thought they are the same as our original one. This means that we overcounted each coloring exactly 8 times, so the number of colorings with two red faces on the opposite sides is $\frac{24}{8} = 3$.



Top view of 8 versions of the same coloring.

Here is another way to see this. If two opposite faces are colored red, then we can always rotate the cube on the table so that it rests on a red face and the blue side faces us. Now if somebody tells us the color of the face opposite of the blue one then we have the full coloring, as we can always flip the other two faces into each other (with keeping a red face on the bottom, and the blue side facing us). Since there are three choices for the color opposite of the blue side, we have three colorings with two red faces on the opposite sides.

In the second case we can start similarly, but now we color two adjacent faces red. Imagine that we orient the cube so that the two red faces are pointing up. We label the remaining faces with A, B, C, D and assign the remaining four colors as before, 24 different ways. But we again overcounted: we can always rotate the two red faces into each other, so each coloring was counted twice. This gives $\frac{24}{2} = 12$ colorings.

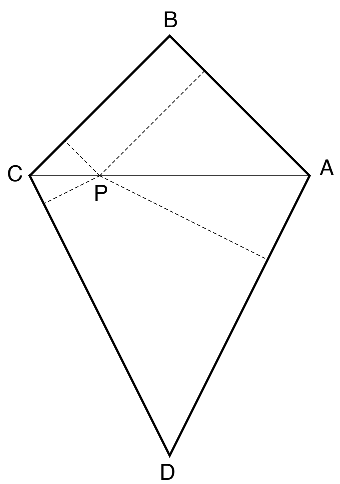


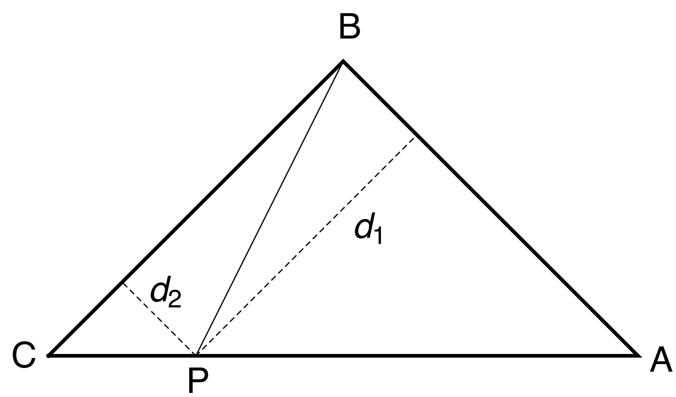
Overall we have $3 + 12 = 15$ colorings with two red faces, and hence there are $5 \cdot 15 = 75$ ways to color the cube according to the specifications.

3. A quadrilateral $ABCD$ has side lengths $AB = BC = 3$ and $CD = DA = 5$. Let P be a point on the line segment AC and compute the distance of P from each of the four sides. Show that the sum of these four distances does not depend on the position of P.

SOLUTION. The triangle ABC and the triangle ADC are both isosceles as we have $AB = BC$ and $CD = AD$. Denote the distance of the point P from the sides AB, BC, CD, DA by d_1, d_2, d_3 and d_4, respectively. We need to show that the sum $d_1 + d_2 + d_3 + d_4$ does not depend on the position of P on AC. We will show that each of the sums $d_1 + d_2$ and $d_3 + d_4$ separately do not depend on the position of P.

Consider the area of the triangle APB. We can compute this as $\frac{AB \cdot d_1}{2} = \frac{3}{2} d_1$ as d_1 is exactly the height corresponding to the side AB. Similarly, the area of the triangle BPC is $\frac{BC \cdot d_2}{2} = \frac{3}{2} d_2$. The triangle ABC is the union of these two triangles, so the area of ABC is the sum of the two areas, which is $\frac{3}{2}(d_1 + d_2)$. But this means that $d_1 + d_2 = \frac{2}{3}\text{area}(ABC)$, which does not depend on the position of P. We can similarly show that $d_3 + d_4$ is always $\frac{2}{5}$ times the area of the triangle ADC. This shows that $d_1 + d_2 + d_3 + d_4$ indeed does not depend on the position of P.





4. How many positive real numbers x satisfy $\frac{\sqrt{x+2017}}{100} = x - \lfloor x \rfloor$? Here $\lfloor x \rfloor$ refers to the greatest integer less than or equal to x. (E.g. $\lfloor 3.9 \rfloor = 3$ and $\lfloor 5 \rfloor = 5$.)

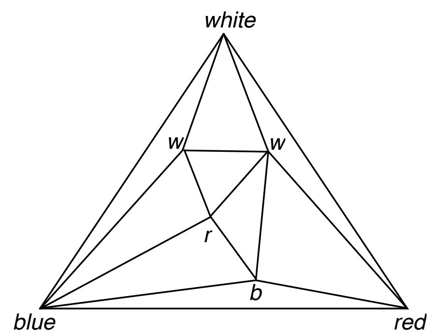
SOLUTION. We show that there are 7982 such values.

Let k be an integer with $0 \le k \le 7981$. For x in the interval $[k, k+1)$, $x - \lfloor x \rfloor = x - k$. The original equation is satisfied at x in the interval $[k, k+1)$ exactly when $x - k - \frac{\sqrt{x+2017}}{100} = 0$.

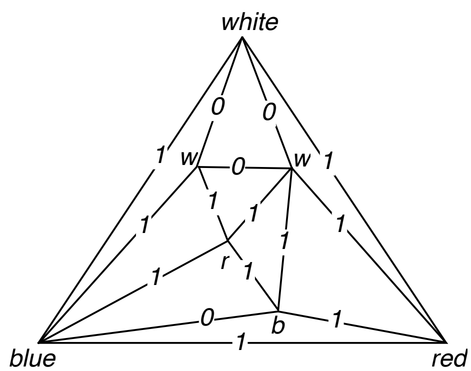
Letting y be the positive number such that $x + 2017 = y^2$, we see that $x - k - \frac{\sqrt{x+2017}}{100}$ is the same as $(y^2 - 2017 - k) - \frac{y}{100}$, and while x is in the interval $[k, k+1)$, y is in the interval $\left[\sqrt{k+2017}, \sqrt{k+2018}\right)$. Also, $(y^2 - 2017 - k) - \frac{y}{100}$ is a quadratic polynomial whose graph is a parabola with a vertex at $y = \frac{1}{200} < \sqrt{2017}$ implying that $(y^2 - 2017 - k) - \frac{y}{100}$ is a strictly increasing function on $\left[\sqrt{k+2017}, \sqrt{k+2018}\right)$. When we plug in $y = \sqrt{k+2017}$, we get $-\frac{\sqrt{k+2017}}{100} < 0$, and when we plug in $y = \sqrt{k+2018}$, we get $1 - \frac{\sqrt{k+2018}}{100} \ge 1 - \frac{\sqrt{9999}}{100} > 0$. Thus our parabola is strictly increasing on the interval $\left[\sqrt{k+2017}, \sqrt{k+2018}\right)$, it is negative at the beginning and positive at the end, so it must have exactly one zero in between. This shows that for each integer k with $0 \le k \le 7981$, we have exactly one solution x in $[k, k+1)$.

If $k \ge 7982$, then most of the previous argument can be repeated, but now when we plug in $y = \sqrt{k+2018}$ into $(y^2 - 2017 - k) - \frac{y}{100}$, we get $1 - \frac{\sqrt{k+2018}}{100} \le 1 - \frac{\sqrt{10000}}{100} \le 0$. Thus our parabola is strictly increasing on the interval $\left[\sqrt{k+2017}, \sqrt{k+2018}\right)$, it is negative at the beginning and nonpositive at $\sqrt{k+2018}$, so it cannot have a solution in the interval $\left[\sqrt{k+2017}, \sqrt{k+2018}\right)$. This shows that there are no solutions x with $x \ge 7982$, which means that the 7982 solutions found in $[0, 7982)$ give all of the positive solutions.

5. The United Triangular States is a triangle shaped country with a red flag, a white flag, and a blue flag marking its three corners. The country is divided into a number of triangle shaped states. No vertex of any state is on another state's border or on the country's border unless it is also a vertex of that other state or the country. (See the figure for an example.) Every vertex is marked with a red, white, or blue flag. Prove that no matter how the map is drawn and how the flag colors are assigned, there must be at least one state whose three vertices are marked with all three colors.



SOLUTION. Label each of the three sides of every state with a 0 if the two flags at the ends of the side are of the same color and 1 if they have different colors.



Assign to each state a score which is the sum of the three labels on its three sides. Then a state can have a score of 0 (if all three of its flags are of the same color), 2 (if its flags are of two colors) or 3 (if its flags are 3 different colors).

Now consider the sum of the scores for all the states. If the side of a state is inside of the country, then it is a side shared by exactly two states, and if the side is the side of the country, then it is only a side of one state. Thus, when we add up the scores of the states, each 'inside' label will appear twice, and each of the three outer boundary labels (which are all 1) appear once. This means that the sum of the scores is always odd (as the contribution of the inside edges is an even number, and the contribution of three outer edges is 3). If the sum of the scores is odd, then there must be at least one state with an odd score. But a state with an odd score must have a score of 3, which means that it has 3 flags with different colors.

Remark: It was important to assume that there is no state with a vertex which is along the side of another state. Otherwise a side could belong to more than two states, and our proof wouldn't have worked. In fact, if we do not make this assumption, then we can come up with a counterexample, as in the picture below.