WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET I (2017-2018)

1. During a certain calendar year there were three months in a row so that each month had exactly 4 Saturdays. Find all possible choices for three consecutive months in which this can happen. (Make sure you justify why you have found all the possibilities.)

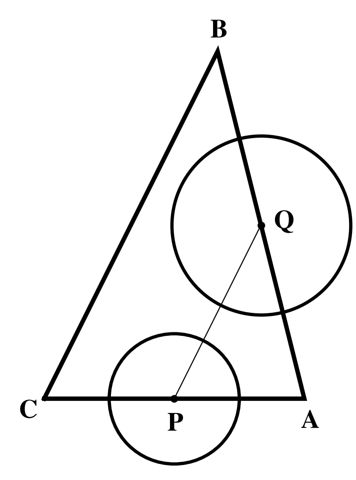
SOLUTION. In 91 consecutive days, since $91 = 13 \times 7$, there are 13 of each day of the week, including Saturday. If these 91 days fall within 3 consecutive months, then one of those months must have at least 5 Saturdays (since if each month had at most 4, there would be at most 12 Saturdays total). In March through December, there are no two consecutive months that each have 30 days. So in March through December, any 3 consecutive months have at least 91 days (since the first two have at least 61 combined and the third has at least 30), and thus the conditions of the problem cannot occur. However, in a non-leap year, if January starts on a Sunday, there will be Saturdays on January 7, 14, 21, and 28, then February 4, 11, 18, 25, and March 4, 11, 18, 25, so January-March is possible (2017 was such a year). We see that February-April is also possible, if for example, February starts on a Sunday in a non-leap year. Then there are Saturdays on February 7, 14, 21, 28, and March 7, 14, 21, 28 and April 4, 11, 18, 25. (This happened in 2015.)

2. Find 2017 positive integers (not necessarily distinct), so that their sum is the same as their product.

SOLUTION. Trying examples, we quickly realize that for most examples the product is much bigger than the sum. To avoid this, we will consider an example with many 1's. Suppose we have 2015 1's and an a and a b. The sum of these numbers is $2015+a+b$ and the product is ab, so they will satisfy the requirements if and only if $2015+a+b = ab$. We can rearrange this equality to get $2016 = ab-a-b+1 = (a-1)(b-1)$. Choosing $a = 2$ and $b = 2017$ we have $(a-1)(b-1) = 2016$. So we see that we can take 2015 1's and a 2 and a 2017. They have sum and product 2×2017. (There are a number of other possible solutions.)

3. On each side of a triangle construct a circle where the center of the circle is the midpoint of the side, and the radius of the circle is $\frac{1}{4}$ the length of the side. Show that if for each pair of two circles there is at least one point common to both circles, then the triangle must be an equilateral triangle.

SOLUTION. Let the vertices of the triangle be A, B, and C, and suppose that the longest side is $\overline{BC}$ which has length a. Let sides $\overline{AC}$ and $\overline{AB}$ have midpoints P and Q and side lengths b and c, respectively. The circle centered at P has radius $\frac{b}{4}$, and the circle centered at Q has radius $\frac{c}{4}$, and because these two circles have at least one point in common, $PQ \le \frac{b+c}{4}$. The $\triangle AQP$ is similar to $\triangle ABC$ as $AQ = \frac{1}{2}AB$, $AP = \frac{1}{2}AC$ and $\angle PAQ = \angle CAB$. Because of this we have $\frac{PQ}{BC} = \frac{AQ}{AB} = \frac{1}{2}$, and thus $PQ = \frac{a}{2}$. This shows that $\frac{a}{2} \le \frac{b+c}{4}$ and $2a \le b+c$. But $a \ge b$ and $a \ge c$, so $2a \le b+c \le 2a$ implying that $2a = b+c$. This means that in both $a \ge b$ and $a \ge c$ we must have equality, and we must have $a = b = c$. This proves that $\triangle ABC$ is an equilateral triangle.



4. 10 players play a round robin tournament. This means that each player plays with every other player exactly once. In each game there is a winner and a loser, the winner receives one point while the loser does not get any points. At the end of the tournament we list the number of points for each player, and compute the sum of the squares of these 10 numbers. Show that the number we got this way is at most 285.

SOLUTION. We first show how we can achieve 285 (although this was not part of the problem). Suppose that the players are numbered by 1,2, ..., 10, and in each game the player with the higher number wins. Then Player 1 will have 0 points, Player 2 will have 1 point, Player 3 will have 2 points, and so on. The sum of the squares of the points is exactly

$$0^2 + 1^2 + 2^2 + \cdots + 9^2 = 285.$$

Now we will show that the sum of the squares cannot be larger than 285. There are finitely many ways that the tournament can end. Imagine that we evaluate the sum of the squares of the scores for each outcome, and pick the outcome where this value is the largest possible. (There might be several outcomes with the maximal value, in that case we just pick one of them.) We will call this outcome the 'maximal configuration'.

We first show that in the maximal configuration there cannot be two players with the same score. We prove this by contradiction: assume that there are two players A and B so that they both have p points at the end of the tournament. We can assume that in the game involving A and B the winner was A. (Otherwise we can just switch their names.) Now imagine the outcome where we change the result of the game between A and B (so that B will win), but we keep the results of all the other games the same. Then A lost a point, B gained a point, so A will have $p-1$ points and B will have $p+1$ points. The other eight players will have the exact same score since we didn't change the result of any game not involving both A and B. Thus the sum of the squares of the scores changed by

$$(p+1)^2 + (p-1)^2 - (p^2 + p^2) = p^2 + 2p + 1 + p^2 - 2p + 1 - 2p^2 = 2.$$

This means that the sum of the squares in the new, modified configuration is larger than the sum of the squares in the maximal configuration, which contradicts our choice of the maximal configuration. The found contradiction shows that in a maximal configuration all scores are different.

A player can have at most 9 points (since they play 9 games). So if the 10 players all have different scores, then the scores must be 0, 1, 2, ..., 9 and in that case the sum of the squares is 285. This proves that the sum of the squares of the scores cannot be larger than 285.

5. Erin and Mikayla play the following game. They shuffle a regular deck of cards and Erin draws cards one by one until she finds the first black ace. Then Mikayla draws cards from the remaining deck one by one until she finds the remaining other black ace. The winner is the player who drew more cards, and there is a tie if they drew the same number of cards. Is this a fair game? If not, who has a bigger chance to win?
(A regular deck of cards has 52 cards, with two black aces. We assume that after shuffling the cards, all possible configurations are equally likely.)

SOLUTION. We will show that the game is fair: out of all orderings of a deck of cards there are the same number of winning configurations for Erin as for Mikayla.

For a given configuration let a_k denote the card in the kth position. (Thus a_1 is the first card, a_2 is the second, a_{52} is the last one.) Denote by x the position of the first black ace and y the position of the second black ace. Note that we must have $1 \leq x < y \leq 52$. Now imagine that we transform the configuration the following way: we consider the block of cards from position 1 to $x-1$, the block of cards from position $x+1$ to $y-1$ and switch the two blocks:

$$\text{original configuration:} \qquad (\underbrace{a_1, \ldots, a_{x-1}}_{\text{first block}}, \underset{\substack{\uparrow \\ \text{1st black ace}}}{a_x}, \underbrace{a_{x+1}, \ldots, a_{y-1}}_{\text{second block}}, \underset{\substack{\uparrow \\ \text{2nd black ace}}}{a_y}, a_{y+1}, \ldots, a_{52})$$

$$\Downarrow$$

$$\text{transformed configuration:} \qquad (\underbrace{a_{x+1}, \ldots, a_{y-1}}_{\text{first block}}, a_x, \underbrace{a_1, \ldots, a_{x-1}}_{\text{second block}}, a_y, a_{y+1}, \ldots, a_{52})$$

Note that one or both of the blocks might be empty. In the transformed configuration a_x and a_y are still the first and second black ace, respectively, though their positions in the deck might change.

Any configuration where Erin would win (this happens when $x > y - x$) is transformed into a configuration where Mikayla would win, and vice versa. We can also check that performing the transformation once more will bring us back to the original configuration. This means that we can exactly match up the configurations where Erin wins with the configurations where Mikayla wins. Thus the corresponding numbers are the same, which shows that the game is fair.