WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET V (2016-2017)

1. Find all solutions of the following equation:

$$|\cdots|||x|-1|-2|-\cdots|-2017|=0.$$

(On the left side of the equation we start with $|x|$, subtract 1, take absolute values, subtract 2, take absolute values, subtract 3, and so on.)

SOLUTION. The number $x_1 = 1+2+\cdots+2017 = \frac{2016\cdot 2017}{2}$ is a solution, because we have a nonnegative number inside every absolute value on the right side, and

$$x_1 - 1 - 2 - 3 - \cdots - 2017 = 0.$$

If x_1 is a solution, then $x_2 = -(1+\cdots+2017) = -\frac{2016\cdot 2017}{2}$ is also a solution, as $|x_2| = |x_1|$. We will show that there are no other solutions.

Suppose that x is a solution. Let $y_0 = |x|$ and for a positive integer n denote by y_n the number $|\cdots|||x|-1|-2|-\cdots|-n|$. So $y_1 = ||x|-1|$, $y_2 = |||x|-1|-2|$ and so on. Note that for any $k \geq 0$ the number y_k is always nonnegative and $y_{k+1} = |y_k - (k+1)|$.

We have $0 = y_{2017} = |y_{2016} - 2017|$, which tells us that $y_{2016} - 2017 = 0$ and $y_{2016} = 2017$. We also have $2017 = y_{2016} = |y_{2015} - 2016|$, so $y_{2015} - 2016$ is either 2017 or -2017. Since y_{2015} is nonnegative, we get $y_{2015} - 2016 \geq -2016 > -2017$ and hence it can only be equal to 2017. Thus $y_{2015} = 2016 + 2017$. These steps can be repeated again:

$$y_{2015} = 2016 + 2017 = |y_{2014} - 2015| \Rightarrow \pm(2016+2017) = y_{2014} - 2015 \geq -2015$$

which yields $y_{2014} = 2015 + 2016 + 2017$. Repeating this further, for any $k \geq 0$

$$y_k = (k+1) + (k+2) + \cdots + 2017$$

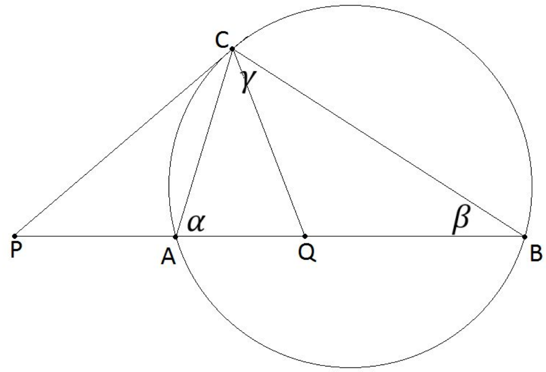
and in particular

$$|x| = 1 + 2 + \cdots + 2017.$$

But this means that x can only be x_1 or x_2.

2. Consider a circle, and three points, A, B, C on the circle. Denote by P the intersection point of the tangent line to the circle at the point C with the line AB. Let Q be the intersection of line AB and the angle bisector of angle $\angle ACB$. Show that $PQ = PC$.

SOLUTION. Let α be the measure of $\angle BAC$. Let β be the measure of $\angle ABC$. Let γ be the measure of $\angle ACB$. Without loss of generality, assume that $\beta < \alpha$. Since ABC is a triangle, we have $\alpha + \beta + \gamma = 180°$. Since $\angle ABC$ and $\angle ACP$ subtend the same arc of the circle, we have $\angle ACP = \angle ABC = \beta$. Since CQ bisects angle ACB, we have $\angle ACQ = \gamma/2$. Thus, since ACQ is a triangle, we have $\angle AQC = 180° - \angle ACQ - \angle QAC = 180° - \gamma/2 - \alpha = \beta + \gamma/2$. Also, $\angle QCP = \angle ACQ + \angle ACP = \gamma/2 + \beta$. So $\angle PQC = \angle AQC = \beta + \gamma/2 = \angle QCP$. Thus $\triangle PQC$ is isosceles, and $PQ = PC$.



3. We have a 2×13 table filled with integers. Each cell contains exactly one number. The numbers in the first row are red and the numbers in the second row are blue. Show that we can choose 3 columns so that both the red numbers and the blue numbers in the chosen columns have integer valued averages.

SOLUTION. Denote the red numbers by $a_1, a_2, \ldots, a_{13}$ and the blue numbers by $b_1, b_2, \ldots, b_{13}$. (So in column k we have the red number a_k and the blue number b_k.) We need to find columns $1 \leq i < j < k \leq 13$ so that $\frac{a_i+a_j+a_k}{3}$ and $\frac{b_i+b_j+b_k}{3}$ are both integers, or in other words: $a_i + a_j + a_k$ and $b_i + b_j + b_k$ are both multiples of 3.

Because we are only interested in divisibility by 3, we can work with the remainders of these numbers, and hence we can assume that each number in the table is either 0, 1 or 2. Then out of the 13 numbers in the first row there must be at least 5 that are the same. (If we have at most four from 0, 1, and 2 then we would have at most 12 numbers.) Choose 5 columns so that the numbers in the first row (the red numbers) are all the same. If we choose any three of these columns, then the sum of the red numbers will be a multiple of 3 (since all three numbers are the same). Thus we just have to make sure that the sum of the blue numbers is a multiple of 3.

We have 5 blue numbers, each one is 0, 1 or 2. If a number appears at least 3 times then we are done: choosing three equal numbers will give a sum which is a multiple of 3. If a number doesn't appear among the 5, then one of the other two must appear at least three times (and we are done). Thus we can assume that each of the three numbers appear at least once, and hence we can choose three numbers out of the five so that they are all different. But then their sum is $0+1+2 = 3$ which is divisible by 3. This shows that we can always choose three columns so that sum of the red numbers and the sum of the blue numbers are both multiples of 3.

4. We have a sequence of integers $a_1, a_2, \ldots$ such that $a_1 = 2$ and $a_{n+1} = a_n^2 - a_n + 1$ for all $n \geq 1$. What is the 1000th digit after the decimal place in the decimal representation of

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{2017}}?$$

SOLUTION. By considering the first few elements of the sequence $2, 3, 7, 43, \cdots$ we notice that at least for small n, we have

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = 1 - \frac{1}{a_{n+1} - 1}.$$

We prove this by induction. We can see it is true when $n = 1$. Suppose that the equation above is true for some n. Then it suffices to show that

$$\frac{1}{a_{n+1}} = \frac{1}{a_{n+1} - 1} - \frac{1}{a_{n+2} - 1},$$

because adding this to both sides of the original equation, we obtain

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{n+1}} = 1 - \frac{1}{a_{n+2} - 1},$$

which would complete our induction. Now, since $a_{n+2} - 1 = a_{n+1}^2 - a_{n+1}$, we see that

$$\frac{1}{a_{n+2} - 1} = \frac{1}{a_{n+1}(a_{n+1} - 1)} = \frac{a_{n+1} - (a_{n+1} - 1)}{a_{n+1}(a_{n+1} - 1)} = \frac{1}{a_{n+1} - 1} - \frac{1}{a_{n+1}},$$

as desired. Thus

$$x := \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{2016}} = 1 - \frac{1}{a_{2017} - 1}.$$

We claim that the 1000th digit after the decimal place of x is 9, which will follows from the stronger claim that $1 - 10^{-1000} < x < 1$. We will show that $10^{1000} < a_{2017} - 1$, and then it follows that

$$1 - 10^{-1000} < 1 - \frac{1}{a_{2017} - 1} < 1.$$

In fact, we will show by induction that for $n \geq 2$, we have $a_n - 1 \geq 2^{2^{n-2}}$. This is true for $n = 2$. If we have $a_n - 1 \geq 2^{2^{n-2}}$ for some n, then we have $a_{n+1} - 1 = a_n(a_n - 1) \geq 2^{2^{n-2}} \cdot 2^{2^{n-2}} = 2^{2^{n-1}}$, which completes the induction. So $a_{2017} - 1 \geq 2^{2^{2015}} = (2^{2^{1995}})^{2^{10}} > (10)^{1000}$.

5. Cheryl chose three distinct integers between 1 and 5. We would like to identify these numbers. We are allowed to make queries in the following form: we give Cheryl 3 distinct numbers a, b, and c, and she tells us how many of these numbers are among chosen ones. For example, if we guess 2, 3, and 4 and the chosen numbers are 1, 2, and 5, we get the answer 1. Find the smallest number of queries needed to guarantee that we can always identify the three chosen numbers. Note that we do not need to actually name the three correct numbers in a query; it is sufficient to identify them without a doubt from the replies to our queries.

SOLUTION. We can always identify Cheryl's numbers with 3 queries, but we cannot always identify Cheryl's numbers with fewer than 3 queries. Denote by A the collection of Cheryl's numbers. The following strategy will always determine the elements of A.

For our first query we guess 1, 2, 3. Cheryl will respond 1, 2, or 3. (There are only 5 numbers, so there must be at least one number that we got right.)

1. If the answer to the 1st query is 3 then we are done, Cheryl's numbers are $1, 2, 3$.
2. If the answer to the 1st query is 1, we know that 4 and 5 are in A. By guessing $1, 4, 5$ and then $2, 4, 5$, we will either get the answer 3 for one of these queries (and then we are done), or we get the answer 2 for both questions (and then we know that Cheryl's numbers are $3, 4, 5$.)

3. If the answer to the first query is 2, exactly one of 4 and 5 is in A.

   For our second query we guess 1, 2, and 4. Cheryl will again respond 1, 2, or 3.

   (a) If the answer to the second query is 3, then we are done.

   (b) If the answer to the second query is 1, we know 3 is in A and 4 is not. So 3 and 5 are in A and 4 is not, so one more query (e.g. $1, 3, 5$) determines if 1 or 2 is in A.

   (c) If the answer to the second query is 2, then when switching from $1, 2, 3$ to $1, 2, 4$ the number of correct numbers does not change. Thus we know that either 3 and 4 are both in A or that neither 3 or 4 is in A.

   For our third query we guess 1, 3, and 4.

   i. If the answer is 1, we know that 3 and 4 are not in A, so $A = \{1, 2, 5\}$.

   ii. If the answer is 2, then 3 and 4 are in A and 1 is not, so $A = \{2, 3, 4\}$.

   iii. If the answer is 3, $A = \{1, 3, 4\}$.

This shows that we can always identify A with at most three queries.

On the other hand, if we first query a, b, and c, and Cheryl gives the answer 1, we know that A contains the two integers in A that are not in $\{a, b, c\}$ and exactly one of a, b, or c. There is no way to guarantee discovering which of a, b, and c is in A in just 1 query because if our query contains just one of a, b, and c, then an answer of 2 still leaves two possible values, and if our query contains two of a, b, and c, then an answer of 2 still leaves two possible values. This shows that 2 queries is not always enough to determine A.