WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET IV (2016-2017)

1. Find all the three-digit positive integers n such that if n is added to the number formed by writing the three digits of n in reverse order, the sum is 1372.

SOLUTION. We will use the notation $\overline{xyz}$ for a three digit number with digits x, y and z in this order.

Let $\overline{abc}$ be a three digit number such that $\overline{abc} + \overline{cba} = 1372$. Then $1372 = (100a + 10b + c) + (100c + 10b + a) = 101(a + c) + 20b$. This gives $101(a + c) = 1372 - 20b$. Since $1372 - 20b = 4(343 - 5b)$, the number $a + c$ must be divisible by 4. Because $1 \leq a + c \leq 18$, the possible choices for $a + c$ are 4, 8, 12 and 16. Out of these choices only $a + c = 12$ gives an integer for $b = \frac{1}{20}\big(1372 - 101(a+c)\big)$ which gives $b = 8$. There are 7 pairs of digits (a, c) such that $a + c = 12$: $(3, 9), (4, 8), (5, 7), (6, 6), (7, 5), (8, 4)$, and $(9, 3)$. This gives 7 possible numbers for n: 389, 488, 587, 686, 785, 884, and 983. One can check that all of these satisfy the requirements.

2. Let $ABCD$ be a square. Find all possible positions of a point R on the segment CD such that there exist points P, Q, and S such that $PQRS$ is a square, and P is on segment AB, and Q is on segment AC, and Q is closer to line BC than any of the points P, R, or S.

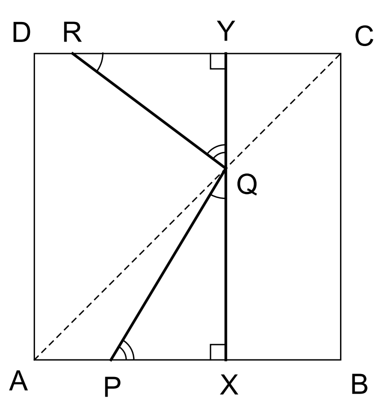
SOLUTION. The only possible location of R is $R = D$.

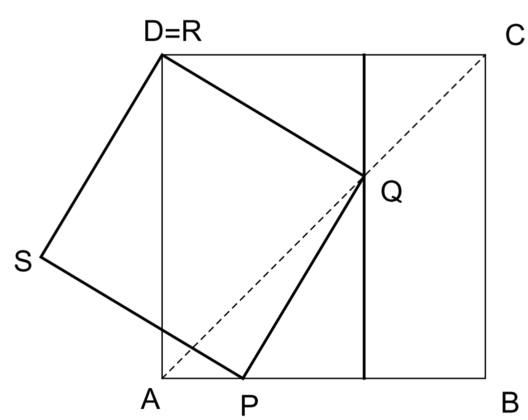
Let the line through Q perpendicular to AB intersect AB at X and CD at Y. Since Q is the closest vertex of $PQRS$ to line BC, we have that P is on segment AX and R is on segment YD.

Since $\angle PQX + \angle PQR + \angle RQY = 180°$ and $\angle PQR = 90°$, we have $\angle PQX + \angle RQY = 90°$. Since $\triangle PXQ$ is a right triangle, we have $\angle PQX + \angle XPQ = 90°$, and so $\angle RQY = \angle XPQ$. We have $\angle PXQ = 90° = \angle QYR$. Since $PQRS$ is a square, we have $PQ = RQ$ and thus $\triangle PXQ$ and $\triangle QYR$ are congruent.

Since $\triangle AQX$ is a $45° - 45° - 90°$ triangle, we have $AX = XQ$. Also, $AX = YD$ since $AXYD$ is a rectangle. Finally $XQ = YR$ because $\triangle PXQ$ and $\triangle QYR$ are congruent. Thus, $YR = YD$. Since R is on segment YD, the given the conditions on the points imply that $R = D$.

One possible configuration is $P = A$, Q is the center of $ABCD$, and $R = D$. This shows that $R = D$ is possible. Moreover, it is not too hard to show (reversing the steps of the previous proof) that for any point P on AB strictly between A and B, we can construct a square $PQDS$ with the given conditions.

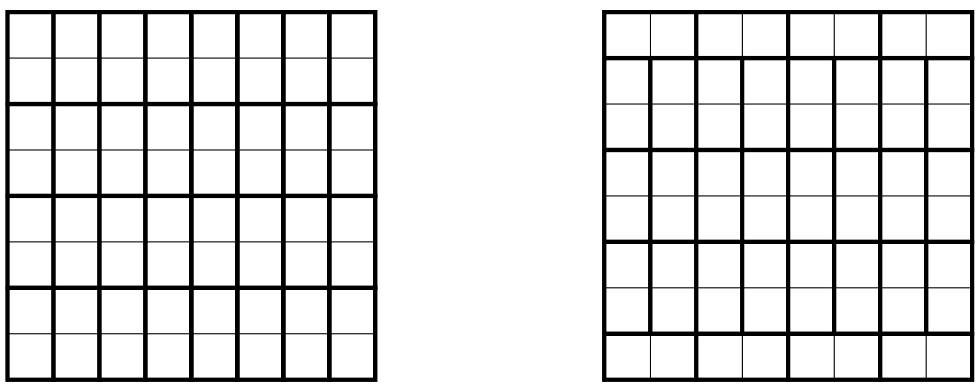




3. We have an 8×8 board consisting of 64 unit squares. We have thirty-two 2×1 tiles which are built from two unit squares. Somebody placed 16 of these tiles on the board so that the tiles do not overlap, and each tile covers exactly two unit squares on the board. Show that we can place an additional (17th) 2×1 tile on the board so that it covers exactly two unit squares without overlapping with any of the other tiles.

SOLUTION. Denote the positions of the 16 tiles by $T_1, T_2, \ldots, T_{16}$. We will show later that we can cover the full board with 32 tiles without overlapping so that one of the 32 tiles is T_1. Denote the tiles in this covering by $T_1, S_1, S_2, \ldots, S_{31}$. Each one of the tiles among $T_2, T_3, \ldots, T_{16}$ can overlap with at most two of the tiles from $S_1, \ldots, S_{31}$, this gives at most 30 tiles. Thus at least one of the tiles among $S_1, \ldots, S_{31}$ will not overlap with any of the tiles $T_2, T_3, \ldots, T_{16}$ and since it cannot overlap with T_1 either (because $T_1, S_1, S_2, \ldots, S_{31}$ is a full cover without overlapping), we can place this tile on the board next to $T_1, T_2, \ldots, T_{16}$ so that it does not overlap with any of those.

The only thing left to prove is that we can cover the board with 32 tiles without overlaps so that one of the tiles is T_1. We may assume that T_1 is a vertical tile, otherwise we just rotate the board by 90°. The following two tilings contain all possible vertical tiles on the board so either the first or the second one will work for a specific choice of T_1.



4. The numbers $a_0, a_1, \ldots, a_{2017}$ are not all zero, and they satisfy the following 2018 inequalities:

$$a_0 \geq 0,\ a_0 + a_1 \geq 0,\ a_0 + a_1 + a_2 \geq 0,\ \ldots,\ a_0 + a_1 + \cdots + a_{2017} \geq 0.$$

Show that there is no real number x greater than 1 that satisfies the equation

$$a_0 x^{2017} + a_1 x^{2016} + \cdots + a_{2016} x + a_{2017} = 0.$$

SOLUTION. We will rewrite the left side of the equation so that the terms a_0, $a_0 + a_1, \ldots$ show up. For a non-negative integer k let s_k denote the sum $a_0 + \cdots + a_k$. Then $a_0 = s_0$, $a_1 = s_1 - s_0$, and in general, for $k > 0$ we have $a_k = s_k - s_{k-1}$. Thus

$$\begin{aligned} &a_0 x^{2017} + a_1 x^{2016} + \cdots + a_{2016} x + a_{2017} \\ &\qquad = s_0 x^{2017} + (s_1 - s_0) x^{2016} + \cdots + (s_{2016} - s_{2015}) x + (s_{2017} - s_{2016}). \end{aligned}$$

Now group the terms with s_0, s_1, and so on:

$$\begin{aligned} &s_0 x^{2017} + (s_1 - s_0) x^{2016} + \cdots + (s_{2016} - s_{2015}) x + (s_{2017} - s_{2016}) \\ &\qquad = s_0 (x^{2017} - x^{2016}) + s_1 (x^{2016} - x^{2015}) + \cdots + s_{2016}(x - 1) + s_{2017}. \end{aligned}$$

In the resulting expression each s_i is either multiplied by a term of the form $x^{j+1} - x^j$ or (in the case of $i = 2017$) by one. Each s_i is nonnegative, and if $x > 1$, then $x^{j+1} - x^j > 0$. Thus if $x > 1$, each term in the last expression is nonnegative. The only way it can be equal to zero is if each term is zero. But this would imply that each and every s_i is zero as well. Since we know that this is not the case, there is no real number x greater than one that will make the expression equal to zero.

5. Consider the collection of all the ordered pairs of sets (A, B) satisfying $A \subseteq B \subseteq \{1, 2, 3, \ldots, 2017\}$. Find the average number of elements there are in the B sets of these ordered pairs. (Note that either A or B could be the empty set!)

SOLUTION. More generally, suppose that for some positive integer n, $A \subseteq B \subseteq \{1, 2, 3, \ldots, n\}$. When $n = 1$, there are only 3 ordered pairs to consider: $(\emptyset, \emptyset)$, $(\emptyset, \{1\})$, and $(\{1\}, \{1\})$. The average size of the B sets is $\frac{0+1+1}{3} = \frac{2}{3}$.
When $n = 2$, there are 9 ordered pairs to consider: $(\emptyset, \emptyset)$, $(\emptyset, \{1\})$, $(\emptyset, \{2\})$, $(\emptyset, \{1, 2\})$, $(\{1\}, \{1\})$, $(\{2\}, \{2\})$, $(\{1\}, \{1, 2\})$, $(\{2\}, \{1, 2\})$, and $(\{1, 2\}, \{1, 2\})$. The average size of the B sets is

$$\frac{0 + 1 + 1 + 2 + 1 + 1 + 2 + 2 + 2}{9} = \frac{4}{3}.$$

A careful accounting of the case when $n = 3$ shows that there are 27 ordered pairs, and the average size of the B sets is 2. This leads to the conjecture that for the nth case, there are 3^n ordered pairs, and the average size of the B sets is $\frac{2}{3}n$. We will prove this statement by mathematical induction.

We checked that the statement is true for $n = 1$. Now assume that it is true for some $n = k > 0$. Consider the ordered pairs that occur in the $n = k + 1$ case. There are three types of ordered pairs (A, B) where $A \subseteq B \subseteq \{1, 2, 3, \ldots, k, k + 1\}$:
(1) those where $k + 1 \notin A$ and $k + 1 \notin B$,
(2) those where $k + 1 \notin A$ and $k + 1 \in B$, and
(3) those where $k + 1 \in A$ and $k + 1 \in B$.

The ordered pairs in type (1) are exactly the ordered pairs that occurred in the case when $n = k$. The ordered pairs in type (2) are exactly the ordered pairs that occurred in the case when $n = k$ except that the element $k + 1$ has been added to the B set. The ordered pairs in type (3) are exactly the ordered pairs that occurred in the case when $n = k$ except that the element $k + 1$ has been added to both the A and B sets.

It follows that in the $n = k + 1$ case, there are 3 times as many ordered pairs as there were in the $n = k$ case, that is $3 \cdot 3^k = 3^{k+1}$ ordered pairs. It also follows that the average size of the B sets in the $k + 1$ case is given by

$$\frac{3^k \cdot \frac{2}{3}k + 3^k \cdot \left(\frac{2}{3}k + 1\right) + 3^k \cdot \left(\frac{2}{3}k + 1\right)}{3^{k+1}} = \frac{2}{3}k + \frac{2}{3} = \frac{2}{3}(k + 1)$$

(using the assumption that the average size for k was $\frac{2}{3}k$). This proves our conjecture by mathematical induction. Hence when $n = 2017$, the average size of the B sets is $\frac{2}{3} \cdot 2017 = 1344\frac{1}{3}$.