

WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH
SOLUTIONS TO PROBLEM SET III (2016-2017)

1. (a) Let $a, b, c,$ and d be integers for which $ad \neq bc$. Show that it is always possible to write the fraction $\frac{1}{(ax+b)(cx+d)}$ in the form $\frac{r}{ax+b} + \frac{s}{cx+d}$ where r and s are fractions.
 (b) Write the following sum as a common fraction:

$$\frac{1}{1 \cdot 4} + \frac{1}{4 \cdot 7} + \frac{1}{7 \cdot 10} + \cdots + \frac{1}{2998 \cdot 3001}.$$

SOLUTION. We want r and s such that $\frac{r(cx+d)+s(ax+b)}{(ax+b)(cx+d)} = \frac{1}{(ax+b)(cx+d)}$. If $rc + sa = 0$ and $rd + sb = 1$, this will hold. Solving these equations gives $r = a/(ad - bc)$ and $s = c/(bc - ad)$, and we can check these values satisfy $rc + sa = 0$ and $rd + sb = 1$.

Motivated by part (a), we write

$$\frac{1}{(3k+1)(3(k+1)+1)} = \frac{1/3}{(3k+1)} - \frac{1/3}{(3(k+1)+1)}.$$

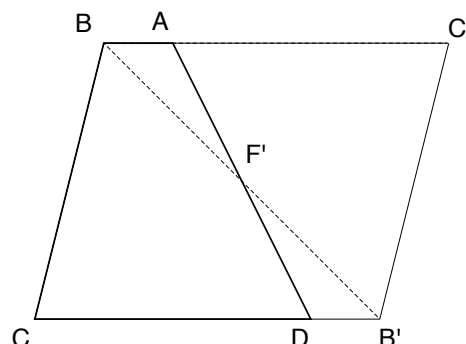
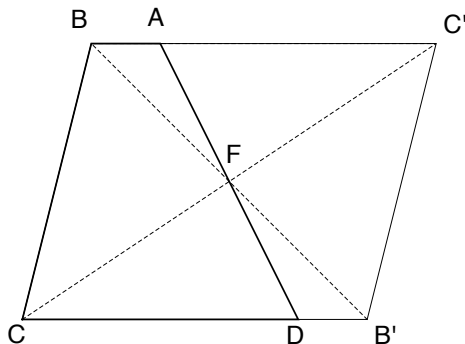
Then we see that the desired sum is

$$\left(\frac{1/3}{1} - \frac{1/3}{4}\right) + \left(\frac{1/3}{4} - \frac{1/3}{7}\right) + \left(\frac{1/3}{7} - \frac{1/3}{10}\right) + \cdots + \left(\frac{1/3}{2998} - \frac{1/3}{3001}\right) = \frac{1}{3} \left(1 - \frac{1}{3001}\right) = \frac{1000}{3001}.$$

2. In the trapezoid $ABCD$ the sides AB and CD are parallel and $BC = AB + CD$. Let F be the midpoint of AD . Find the angle BFC .

SOLUTION. Rotate the trapezoid 180° about the point F . Since F is a midpoint of AD , the point A will go to D and the point D goes to A . Denote the image of B with B' and the image of C by C' . Since $AB \parallel CD$, the angles BAD and CDA add up to 180° , and this means that $B, A,$ and C' will be on the same line, and the same holds for C, D and B' . The rotation takes BC into a parallel line: $BC \parallel C'B'$. This implies that the quadrilateral $BCB'C'$ is a parallelogram. Moreover, since $AC' = CD$ by the rotation, we have $BC' = AB + CD = BC$, which shows that $BCB'C'$ is actually a rhombus.

The diagonals of a rhombus are perpendicular. We will show that the diagonals BB' and CC' intersect in the point F , from this it follows that the angle BFC is 90° . Denote the intersection of BB' and AD by F' . The triangles ABF' and $DB'F'$ are at F' as well, which means that the intersection of the diagonals is indeed F .



3. Find all pairs of integers m, n which solve the equation $\sqrt{n + \sqrt{n + \sqrt{n}}} = m$.

SOLUTION. If m and n solve the equation, then they both have to be nonnegative. If $m = 0$, then n must be zero, and this gives a solution to the equation. We will show that there are no solutions with a positive m and n .

Suppose that n and m are positive integer solutions of the equation. Squaring both sides gives $n + \sqrt{n + \sqrt{n}} = m^2$. That means that $\sqrt{n + \sqrt{n}} = m^2 - n$ is also an integer. Squaring this again gives $n + \sqrt{n} = (m^2 - n)^2$ which tells us that \sqrt{n} is also an integer. Let us call this integer k which is also positive. Then $(m^2 - n)^2 = n + \sqrt{n} = k^2 + k$ and thus $k^2 + k$ is a perfect square. But if k is positive then

$$k^2 < k^2 + k < k^2 + 2k + 1 = (k + 1)^2,$$

which means that $k^2 + k$ is strictly between two consecutive perfect squares. That means that $k^2 + k$ cannot be a perfect square, and the resulting contradiction shows that the only integer solution to our equation is $n = m = 0$.

4. Find the maximal number of integers we can choose out of the numbers $1, 2, \dots, 50$ so that their product is not divisible by 36. Make sure to prove that your choice is maximal!

SOLUTION. We will show that you can choose 35 numbers so that their product is not divisible by 36, but any 36 will have a product that is divisible by 36.

Suppose that we choose all the numbers not divisible by 3 and then the number 3. This gives 35 numbers, and since the product is not divisible by 9, it will not be divisible by 36 either.

Now suppose that we choose 36 numbers. Since there are 25 even integers up to 50, we must have chosen at least two of them (in fact, at least nine of them). There are 34 integers not divisible by 3 up to 50, so out of the 36 integers we must have at least 2 that are multiples of 3. Thus out of the 36 numbers we have at least two even numbers and at least two multiples of three, which will make the product divisible by 36.

5. We roll a fair die repeatedly and add up the numbers. We stop as soon as when we get to a number that is bigger than 1000. The possible numbers when we stop are 1001, 1002, \dots , 1006. Which one of these has the highest probability?

SOLUTION. The number 1001 has the highest probability.

Because we have a fair die, the probability of rolling any given number between 1 and 6 is the same ($1/6$). Similarly, if we roll the die five times, any sequence of five numbers between 1 and 6 will have the same chance: getting 1, 1, 1, 1, 1 has the same probability of getting 6, 4, 6, 2, 3.

Let us call a sequence of rolls an (n, k) sequence if

- $1 \leq k \leq 6$,
- the number of rolls in the sequence is n ,
- the sum of the numbers is $1000 + k$, and
- the sum of the numbers without the last term is at most 1000.

The probability of stopping at $1000+k$ is just the sum of the probabilities of all (n, k) sequences. To show that it is more probable to stop at 1001 than $1000+k$ with a given $2 \leq k \leq 6$ it is sufficient to check that, for each n , the number of $(n, 1)$ sequences is at least as large as the number of (n, k) sequences, and there is at least one n where the number of $(n, 1)$ sequences is strictly greater than the number of (n, k) sequences.

Suppose that $2 \leq k \leq 6$ is a fixed number and d_1, d_2, \dots, d_n is an (n, k) sequence. (Here d_ℓ is the ℓ^{th} die roll.) Then the very last die roll (d_n) has to be at least k otherwise the sum of the first $n-1$ numbers is bigger than 1000. Now consider the sequence of rolls $d_1, d_2, \dots, d_{n-1}, d_n - (k-1)$. We changed the last roll from d_n to $d_n - (k-1)$, this means that the last number is less than 6. This is an $(n, 1)$ sequence (the sum is 1001, and the sum without the last term is at most 1000), and it has the same probability of happening as our original (n, k) sequence. Moreover, different original sequences give different modified sequences. This shows the number of (n, k) sequences for a given n and $2 \leq k \leq 6$ the number of (n, k) sequences is at most as big as the number of $(n, 1)$ sequences ending with a number less than 6.

We can find $(n, 1)$ sequences ending with a 6, for example with $n = 167$ we can take a 5 and followed by 166 copies 6. This example shows that for $n = 167$ the number of $(n, 1)$ sequences is strictly larger than the number of (n, k) sequences (for $2 \leq k \leq 6$). This proves that 1001 is more probable than $1000+k$ for any $2 \leq k \leq 6$.