WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET II (2016-2017)

1. We have a list of positive integers that add up to 1000. What is the largest possible value of the product of these numbers?

SOLUTION. We will show that the largest possible value of the product is $3^{332} \cdot 2^2$. This can be achieved if we have 332 threes and 2 twos on the list. Now we just have to show that we cannot get a larger number as a product.

Since there are finitely many lists with positive integers adding up to 1000, we will have finitely many possibilities for the possible products as well. That means that there must be a largest value for the product. Call a list *maximal* if it produces the largest product. (There might be several maximal lists.)

If the number 1 appears on the list, then we must have at least one other number a on the list, too. Erasing 1 and a and replacing them with $a+1$ will not change the sum, but will increase the product of numbers: $a \cdot 1 \leq a + 1$. This means that we cannot have the number 1 on a maximal list.

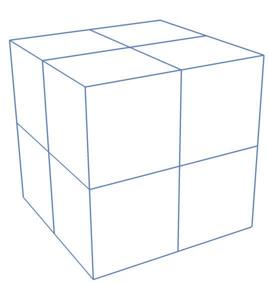
If we have a number a bigger than 4 on the list, then replacing this number with 2 and $a-2$ will keep the sum the same and will increase the product. This is true because $a < 2(a-2) = 2a - 4$ if $4 < a$. Thus a maximal list cannot have a number bigger than 4.

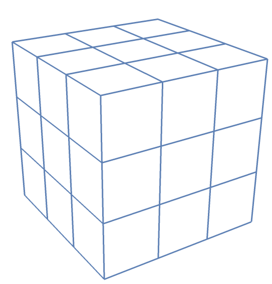
This means that a maximal list can only have the numbers 2, 3 and 4 on it. If a maximal list has the number 4 on it, then replacing a 4 with two copies of 2 will not change the sum and the product. Thus there must be a maximal list with only the numbers 2 and 3 on it.

Finally, a maximal list cannot have more than two copies of the number 2,, as replacing three twos with two threes will not change the sum, but it will increase the product $(2 \cdot 2 \cdot 2 < 3 \cdot 3)$. That means that there is a maximal list with only twos and threes, so that the number of twos is at most 2. Since 1000 is not divisible by 3, we must have at least one 2 on this maximal list, and since $1000 - 2$ is not divisible by three either, the number of twos has to be 2. But then this maximal list has 2 twos and $\frac{1000-2\cdot 2}{3} = 332$ threes, which shows that the maximal product is $3^{332} \cdot 2^2$.

2. Is it possible to cut a cube into 2016 (not necessarily identical) smaller cubes?

SOLUTION. Yes, this can be done in many ways. First note that any cube can be cut into 8 identical smaller cubes by passing three planes through the center of the cube parallel to the sides of the cube. Similarly, any cube can be cut into 27 identical smaller cubes resulting in the familiar $3 \times 3 \times 3$ configuration of a Rubik's cube.





Thus, if you can cut a cube into n smaller cubes, you can replace any one of those n smaller cubes with 8 cubes or with 27 cubes showing that a cube can be cut into $n+7$ or $n+26$ smaller cubes. We begin with $n = 1$ cube, and we would like to end up with $1 + 2015$ cubes. If we can find a positive integers a and b so that $2015 = 7a + 26b$, then by increasing the number of cubes by 7 a times and then increasing this number by 26 b times we will get exactly 2016 cubes.

It is not hard to find such a and b numbers: for example we can try small values of b to see if $2015 - 26b$ is divisible by 7. One can check that $2015 - 26 \cdot 4 = 1911 = 7 \cdot 273$, which means that $a = 273$ and $b = 4$ works.

3. Prove that $2^{(2^{(2^2)})} - 2^{(2^2)}$ divides $n^{(2^{(2^2)})} - n^{(2^2)}$ for all positive integer n.

SOLUTION. We have $2^{(2^{(2^2)})} - 2^{(2^2)} = 2^{16} - 2^4 = 65520 = 2^4 \times 3^2 \times 5 \times 7 \times 13$. We have $n^{(2^{(2^2)})} - n^{2^2} = n^{16} - n^4 = n^4(n^{12} - 1)$. We can check that each of $16, 9, 5, 7, 13$ divide $n^4(n^{12} - 1)$ by checking all possibilities for n modulo each of these numbers (i.e. considering the remainders when dividing by these numbers). If n is even, then $2^4 \mid n^4$. Otherwise, $n \equiv \pm 1, \pm 3, \pm 5$, or $\pm 7 \pmod{16}$. So $n^2 \equiv 1$ or $9 \pmod{16}$ and $n^4 \equiv 1 \pmod{16}$. Thus $n^{12} - 1 \equiv 0 \pmod{16}$. If $3 \mid n$, then $3^2 \mid n^4$. Otherwise, $n \equiv \pm 1, \pm 2$ or $\pm 4 \pmod 9$. So $n^2 \equiv 1, 4$ or $7 \pmod 9$ and then $n^6 \equiv 1 \pmod 9$. Thus $n^{12} - 1 \equiv 0 \pmod 9$. We have $n \equiv 0, \pm 1$, or $\pm 2 \pmod 5$. So $n^2 \equiv 0$ or $\pm 1 \pmod 5$. So $n^4 \equiv 0$ or $1 \pmod 5$ and $n^4(n^{12} - 1) \equiv 0 \pmod 5$. We have $n \equiv 0, \pm 1, \pm 2$ or $\pm 3 \pmod 7$. So $n^2 \equiv 0, 1, 4$ or $2 \pmod 7$ and $n^6 \equiv 0$ or $1 \pmod 7$. Thus $n^4(n^{12} - 1) \equiv 0 \pmod 7$. Finally, we have $n \equiv 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$ or $\pm 6 \pmod{13}$. So $n^2 \equiv 0, \pm 1, \pm 4$, or $\pm 3 \pmod{13}$ and $n^4 \equiv 0, 1, 3$, or $9 \pmod{13}$. Thus $n^{12} \equiv 0$ or $1 \pmod{13}$ and $n^4(n^{12} - 1) \equiv 0 \pmod{13}$.

4. We have an 8 feet by 5 feet room. We divide it up into an 8×5 grid of 40 squares with side length 1 foot. We would like to cover the room with 13 rectangular tiles of size 3 feet by 1 foot and an additional square shaped tile of one foot by one foot. (Each tile must be aligned with our grid and tiles cannot overlap with each other.) Determine which, if any, locations in the room we can place the square tile so that we can cover the rest of the room with the 13 rectangular tiles.

SOLUTION. Suppose that we number the squares of the grid with the numbers 1, 2, 3 according to the following scheme:

3	1	2	3	1	2	3	1
1	2	3	1	2	3	1	2
2	3	1	2	3	1	2	3
3	1	2	3	1	2	3	1
1	2	3	1	2	3	1	2

If we place a 3 by 1 tile on the grid, then it will cover exactly one square labeled by 1 2 and 3. Thus if we can place 13 non-overlapping tiles then they will cover 13 of each of the three numbers. We can check that there are 14 1s, 13 2s, and 13 3s in the grid. That means that if a tiling is possible with 13 non-overlapping 3X1 tiles, then the remaining uncovered single square has to be labeled with 1, meaning that it has to be one of the following squares:

	1			1			1
1			1			1	
		1			1		
	1			1			1
1			1			1	

(∗)

The same way, if we change the labels by reflecting them horizontally, we get that the remaining uncovered single square has to be on one of the squares labeled with 1 in the table below:

1	3	2	1	3	2	1	3
2	1	3	2	1	3	2	1
3	2	1	3	2	1	3	2
1	3	2	1	3	2	1	3
2	1	3	2	1	3	2	1

$\Longrightarrow$

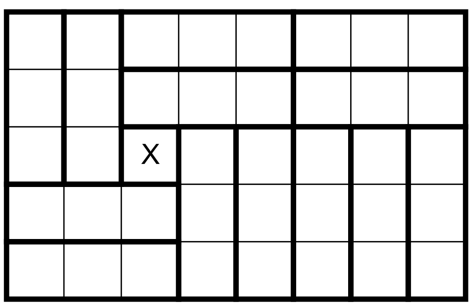
1			1			1	
	1			1			1
		1			1		
1			1			1	
	1			1			1

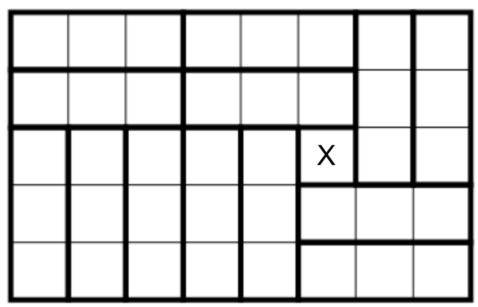
$(**)$

Thus the unused square has to be labeled by a 1 in both the $(*)$ and the $(**)$ arrangement which means that it has to be one of the two squares labeled by X below:

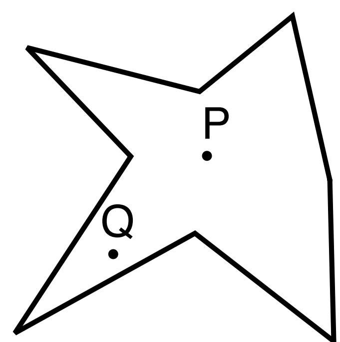
		X			X		

Now we just have to check whether there are tilings corresponding to these two squares. The following examples work:





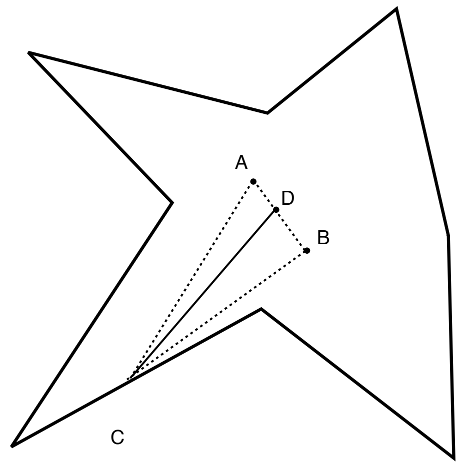
5. An irregularly shaped flat yard is surrounded by a tall fence built from connected straight pieces. We call a location *sightful* if it is inside the yard and a surveyor (shorter than the fence) can see every part of the inside of the fence from that location. (The surveyor is allowed to turn around on the spot). If A is a sightful location and B is a sightful location, show that every point on the straight path from A to B is sightful. (In the figure the point P is sightful, while the point Q is not.)



SOLUTION. Let C be any spot on the fence. Since A is sightful, segment AC is inside the yard (and not crossed by the fence). Similarly BC is inside the yard and not crossed by the fence.

We first show that the segment AB is fully in the yard. Assume the opposite, in that case AB would cross the fence. Since A and B are both inside the yard, this would imply that AB crosses the fence at least twice. But then from A we would not be able to see the part of the fence crossing AB other than the closest.

So we conclude segment AB is entirely within the yard. Thus, triangle ABC is entirely in the yard. But then any point D on segment AB it is in the yard, and the segment DC is entirely in the yard as well. Since this held for any spot C on the fence. we conclude that any point on AB is sightful.


A
D
B
C