WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET I (2016-2017)

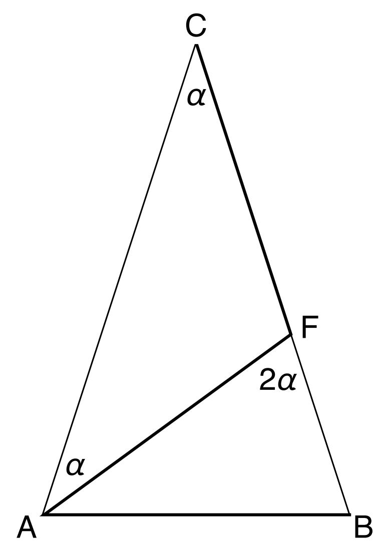
1. There is a bag with marbles in it. Here are some statements about the bag.
(a) There is a blue marble in the bag.
(b) There is a red marble and a blue marble in the bag.
(c) There is a red marble and a blue marble and a green marble in the bag.
(d) There is a red marble in the bag.
Is it possible that exactly two of these statements are true?

SOLUTION. If (c) is true, then (a), (b), and (d) are all true, so all four of the statements are true. So it remains to consider the case when (c) is false. Then, if (b) is true, then (a) and (d) are true, so there are exactly 3 true statements. So, it remains to consider the case when (c) and (b) are false. For there to be exactly 2 true statements in this case, we would need (a) and (d) to be true, but that would imply that (b) is true (and we are considering the case when (b) is false). Thus, having considered all possible cases, we conclude it cannot be the case that exactly 2 of the statements are true.

2. In the triangle ABC we have $AC = BC$. There is a point F on the side BC for which we have $AB = AF = FC$. Find the angles of the triangle ABC.

SOLUTION. Denote the angle $\angle ACB$ by α. Since $AC = BC$, the triangle ABC is isosceles and we have $\angle CAB = \angle CBA = \frac{180^{\circ}-\alpha}{2} = 90^{\circ} - \frac{\alpha}{2}$. The triangle AFC is also isosceles, and thus $\angle CAF = \angle ACF = \angle ACB = \alpha$, and $\angle AFC = 180^{\circ} - 2\alpha$. Then $\angle AFB = 180^{\circ} - \angle AFC = 180^{\circ} - (180^{\circ} - 2\alpha) = 2\alpha$.

Finally, the triangle AFB is also isosceles, so $\angle ABF = \angle AFB = 2\alpha$. But $\angle ABF = \angle ABC = 90^{\circ} - \frac{\alpha}{2}$. This leads to $90^{\circ} - \frac{\alpha}{2} = 2\alpha$ which we can solve to get $\alpha = 36^{\circ}$. Thus the angles of the triangle ABC are $\angle ACB = 36^{\circ}$, $\angle CAB = \angle CBA = 90^{\circ} - \frac{1}{2} \cdot 36^{\circ} = 72^{\circ}$.



3. How many ways can we choose 3 positive integers $a < b < c$ so that $a \cdot b \cdot c$ =30,030?

SOLUTION. We see that $30030 = 2 \times 3 \times 5 \times 7 \times 11 \times 13$. We count the ways to put these 6 primes into 3 sets, corresponding to the prime factors of a, b, and c. There are 3^6 ways to put the 6 primes into each of 3 sets (labelled "first," "second", and "third"), with 3 choices for each of the 6 primes. We then wish to discard the 3 choices where all the primes are put into one set, as that will mean the products of the other two sets are both 1. For the $3^6 - 3$ remaining ways to put the 6 primes into sets, taking the products of the three sets gives 3 distinct integers. Since the 6 primes are all distinct, the only way for two of the products to be the same is if neither contained any primes, and we have eliminated those cases. However, given $a < b < c$ with $abc = 30,030$ there are 6 different configurations of three sets from which it arises, corresponding to the 6 ways of labelling a, b, c as "first," "second", and "third." So the answer is $(3^6 - 3)/6 = 121$.

4. We have a list of 2016 numbers. We notice that if we erase any of the numbers, then the absolute value of the sum of the remaining 2015 numbers is the same as the absolute value of the number we erased. Show that the sum of the 2016 numbers in the list is equal to zero.

SOLUTION. Denote the sum of all 2016 numbers by S, and let x be one of the numbers. By the condition of the problem we have $|x| = |S - x|$. If $x = 0$, then this gives $|S| = 0$ which implies that $S = 0$. (In that case we are done.)

If $x > 0$, then $|x| = x$ and we have $x = |S - x|$. In this case $-x \leq S - x$ which implies $0 \leq S$. If $x < 0$, then $|x| = -x$ and we get $-x = |S - x|$. In that case $S - x \leq -x$ and $S \leq 0$. If we have both a negative and a positive number on the list then $0 \leq S$ and $S \leq 0$ would imply $S = 0$.

If all 2016 numbers are positive then S is positive and $x = S - x$ holds for any x in the list. That would mean that all the numbers on the list are equal to $S/2$. This is a contradiction as in that case the sum of numbers would be equal to $2016 \cdot \frac{S}{2} = 1013 \cdot S > S$. In the same way, we can show that we cannot have 2016 negative numbers on the list. But this means that we either have a zero among the numbers (which implies $S = 0$) or we have at least one positive and at least one negative number on the list which again implies $S = 0$. This completes the proof that the sum of the 2016 numbers of 0.

5. A table with 5 rows and 5 columns is filled with nonnegative integers. With an *add move* we can select one of the 5 rows or one of the 5 columns and add 1 to each number in that row or column. With a *subtract move* we can select one of the 5 rows or one of the 5 columns and subtract 1 from each number in that row or column. Suppose that we can find a sequence of add moves and subtract moves that will reduce our table of nonnegative integers to a table filled with zeros only. Prove that in that case the original table of nonnegative integers can also be reduced to all zeros using *only* subtract moves.

SOLUTION. In any 5×5 table of nonnegative integers, M, and for any $i, j \in \{1, 2, 3, 4, 5\}$, let $m_{i,j}$ refer to the nonnegative integer in the ith row and jth column of the table. (See table on the left.)

$m_{1,1}$	$m_{1,2}$	$m_{1,3}$	$m_{1,4}$	$m_{1,5}$
$m_{2,1}$	$m_{2,2}$	$m_{2,3}$	$m_{2,4}$	$m_{2,5}$
$m_{3,1}$	$m_{3,2}$	$m_{3,3}$	$m_{3,4}$	$m_{3,5}$
$m_{4,1}$	$m_{4,2}$	$m_{4,3}$	$m_{4,4}$	$m_{4,5}$
$m_{5,1}$	$m_{5,2}$	$m_{5,3}$	$m_{5,4}$	$m_{5,5}$

	$m_{i,r}$		$m_{i,s}$	
	$m_{j,r}$		$m_{j,s}$	

We claim that, if i and j are any two row numbers and r and s are any two column numbers, any add move or subtract move applied to M will not change the value of $m_{i,r} + m_{j,s} - m_{i,s} - m_{j,r}$. First note that the values $m_{i,r}, m_{i,s}, m_{j,r}, m_{j,s}$ are at the four corner of a rectangle within the table (see figure above). If we apply an add or a subtract move then we have the following possibilities:
– none of these four numbers change,
– two of these four numbers increase by one, and the two numbers are in the same row or column,
– two of these four numbers decrease by one, and the two numbers are in the same row or column.
We can check, that in all four of these cases the value of $m_{i,r} + m_{j,s} - m_{i,s} - m_{j,r}$ will stay the same.

In an all zero table $m_{i,r} + m_{j,s} - m_{i,s} - m_{j,r} = 0$ (since all the entries are zero). We just checked that $m_{i,r} + m_{j,s} - m_{i,s} - m_{j,r}$ does not change with an add or a subtract move, so if we can reduce our table M to all zeros then we must have $m_{i,r} + m_{j,s} - m_{i,s} - m_{j,r} = 0$ with any choice of i, j, s, r.

But $m_{i,r} + m_{j,s} - m_{i,s} - m_{j,r} = 0$ means that $m_{i,r} - m_{j,r} = m_{i,s} - m_{j,s}$. In other words, if M can be reduced to all zeros by add moves and subtract moves, then for any i and j we can get row i from row j by adding a fixed number to each entry of row j. (The particular number can be positive, negative or zero.) Suppose that the smallest number in the first column is in column k. Then $m_{1,j} - m_{1,k}$ is nonnegative for all j, and applying that many subtract moves to row j will make row j equal to row k. (If $m_{1,j} - m_{1,k} = 0$ then the two rows are already equal.) We can repeat that for each row that is different from row k, which reduces M to a table with 5 identical rows of nonnegative integers. The resulting table consists of 5 constant columns, so a sequence of subtract moves applied to columns of that table can reduce it to a table where all the entries are zeros.

Example: in the first table below the 2nd, 3rd, 4th and 5th rows can be obtained from the first by adding an appropriate number to each entry (2nd: −3, 3rd: 1, 4th: 1 , 5th: −2). The smallest number in the first column is 1 (in the second row), so applying subtract moves to the other rows will reduce the table to a row with constant columns. Finally, we can just apply the subtract moves to the columns to 'zero out' each one.

4	8	6	5	5
1	5	3	2	2
5	9	7	6	6
5	9	7	6	6
2	6	4	3	3

$\Longrightarrow$

1	5	3	2	2
1	5	3	2	2
1	5	3	2	2
1	5	3	2	2
1	5	3	2	2

$\Longrightarrow$

0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0