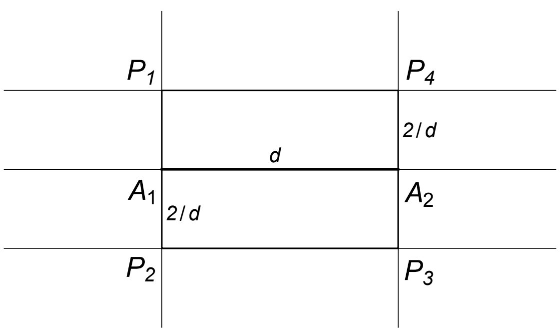
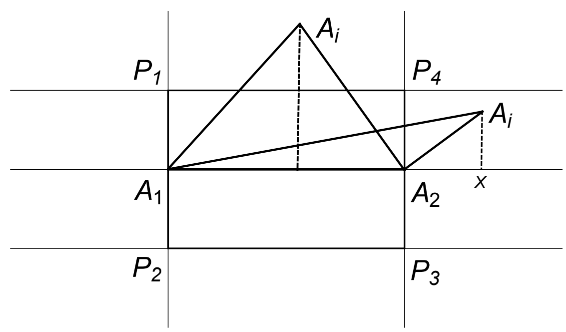
WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET V (2015-2016)

1. There are 2016 points in the plane such that any three of the points are vertices of a triangle with an area of at most 1. Show that we can draw a rectangle of area 4 that contains all the points.

SOLUTION. There are finitely many pairs we can choose out of the 2016 points, so we can choose two points out of the 2016 such that their distance is the largest possible among the points. Denote these points by A_1 and A_2, with the others labeled by $A_3, A_4, \ldots, A_{2016}$. Assume that the length of $\overline{A_1A_2}$ is $d > 0$. Let e and f be lines perpendicular to the line A_1A_2, and let P_1, P_2, P_3 and P_4 be the four points on e and f with distance $\frac{2}{d}$ from the line A_1A_2. (See figure.) The points P_1, P_2, P_3, P_4 are the vertices of a rectangle with area 4. We will show that each of the 2016 points are covered by this rectangle.





Assume that there is a point A_i which is above the line P_1P_4. Then the distance h of A_i from the line A_1A_2 is bigger than $\frac{2}{d}$, and the area of the triangle $A_1A_2A_i$ would be $\frac{1}{2}h \cdot d > \frac{1}{2} \cdot \frac{2}{d} \cdot d = 1$, a contradiction. Similarly, we cannot have a point A_i below the line P_2P_3.

Assume that there is a point A_i which is to the right of line P_3P_4. We will show in this case that $A_1A_i > A_1A_2$, which is a contradiction because A_1A_2 is the largest distance between two of the 2016 points. If A_i is on the line A_1A_2 then $A_1A_i > A_1A_2$, since A_1, A_2, A_i are on the same line and A_2 is between the other two points. If A_i is not on the line A_1A_2 then let X be the point on A_1A_2 for which A_iX is perpendicular to A_1A_2. Then by the Pythagorean Theorem $A_1A_i = \sqrt{(A_iX)^2 + (AA_1)^2} > \sqrt{(A_1X)^2} = A_1X > A_2A_1$, which is again a contradiction. Thus, none of the points lie to the right of P_3P_4. Similarly, we cannot have a point to the left of P_1P_2, and we just proved that all of the points must lie in the rectangle $P_1P_2P_3P_4$.

2. Bill writes down the numbers 1, 2, …, 10 in every possible order. For each possible sequence he circles the numbers whose positions are equal to the number at that place and writes down with red the sum of the circled numbers, writing down 0 if there where none. (E.g. for the sequence $1, 3, 5, 4, 6, 7, 8, 9, 10, 2$ he would circle 1 and 4 and he would write down the number 5 in red.) What is the average of all the red numbers that Bill wrote down?

SOLUTION. There are 10! ways Bill can order the numbers $1, 2, \ldots, 10$. Imagine that Bill writes down these sequences in a table with 10! rows and 10 columns, with one sequence in each row. The sum of the red numbers is the sum of all the circled numbers in the table. We will compute this by considering the sum of the circled numbers in each column.

Consider the numbers circled in the first column. These are exactly the number ones (because if we circle it, it has to have the same value as its position), and there will be exactly 9! of them. This is because there are 9! ways we can fill out the rest of the sequence if we put 1 in the first position. This means that the sum of the circled numbers in the first column is $1 \cdot 9!$.

Consider now the sum of the circled numbers in column i (with $1 \leq i \leq 10$). The numbers that are circled here are exactly the number i, and we will have exactly 9! of them. (This can be shown the same way as before: there are 9! ways to order the remaining 9 numbers if we place i at position i.) Thus the sum of circled numbers in the i^{th} column is $i \cdot 9!$. Thus the sum of all the circled numbers in the table (and the sum of all the red numbers) is $(1+2+\cdots+10) \cdot 9! = 55 \cdot 9!$ and the average is $\frac{55 \cdot 9!}{10!} = \frac{55}{10} = \frac{11}{2}$.

3. Show that if a, b, c are positive numbers with $a+b+c=3$, then

$$\sqrt{2+a^2}+\sqrt{2+b^2}+\sqrt{2+c^2} \geq 3\sqrt{3}.$$

SOLUTION. We first show that for real numbers x, y, z the following inequality is always true:

$$3(x^2+y^2+z^2) \geq (x+y+z)^2.$$

Expanding the right side and rearranging the terms we see that the inequality is equivalent to

$$0 \leq 3x^2+3y^2+3z^2-(x^2+y^2+z^2+2xy+2xz+2yz) = x^2-2xy+y^2+x^2-2xz+z^2+y^2-2yz+z^2.$$

The expression on the right is equal to $(x-y)^2+(x-z)^2+(y-z)^2$. This is always nonnegative, which proves the original inequality.

We apply the inequality with $x=y=1$, $z=a$ to get

$$3(2+a^2) \geq (2+a)^2.$$

Taking square root of both sides (this is valid, since $a>0$), and rearranging gives

$$\sqrt{2+a^2} \geq \frac{1}{\sqrt{3}}(2+a).$$

We can prove the same inequality with b and c. Adding these inequalities up and using $a+b+c=3$ we get

$$\sqrt{2+a^2}+\sqrt{2+b^2}+\sqrt{2+c^2} \geq \frac{1}{\sqrt{3}}(2+a+2+b+2+c) = \frac{9}{\sqrt{3}} = 3\sqrt{3}.$$

Outline of another possible solution: First show that

$$\sqrt{2+a^2}+\sqrt{2+b^2} \geq 2\sqrt{2+\left(\frac{a+b}{2}\right)^2}.$$

You can do that, for example, by squaring both sides, rearranging and squaring again, and showing the resulting (quadratic) inequality. From this inequality we may assume that $a=b$, which means that we have to show

$$\sqrt{2+a^2}+\sqrt{2+a^2}+\sqrt{2+(3-2a)^2} \geq 3\sqrt{3}.$$

Squaring, rearranging and squaring again yields a quadratic inequality which can be shown easily.

4. Show that the number $(2016^2)!$ is divisible by $(2016!)^{2016}$. For an extra point show that it is also divisible by $(2016!)^{2017}$.

SOLUTION. If $b > a$ then there are $\frac{b\cdot(b-1)\cdots(b-a+1)}{a\cdot(a-1)\cdots 1} = \frac{b!}{a!(b-a)!}$ ways we can choose a group of a people from a larger group of size b. The shorthanded notation for the expression $\frac{b!}{a!(b-a)!}$ is the *binomial coefficient* $\binom{b}{a}$ (which is read as 'b choose a'). Since $\binom{b}{a}$ counts the number of ways we can do something, it must be an integer. We will use this idea to tackle the problem.

Let n be a positive integer. In how many ways can we assign n^2 people to n teams with n people on each team? There are $\binom{n^2}{n}$ ways to select n people to be on team 1. Then there are n^2-n people remaining, so there are $\binom{n^2-n}{n}$ ways to select n people to be on team 2. After assigning people to the first k teams, there are n^2-kn people remaining, so there are $\binom{n^2-kn}{n}$ ways to select people to be on team $(k+1)$. Thus, the total number of ways to assign n^2 people to n teams with n people on each team is

$$\binom{n^2}{n}\cdot\binom{n^2-n}{n}\cdot\binom{n^2-2n}{n}\cdots\binom{n}{n} = \frac{(n^2)!}{n!(n^2-n)!}\cdot\frac{(n^2-n)!}{n!(n^2-2n)!}\cdot\frac{(n^2-2n)!}{n!(n^2-3n)!}\cdots\frac{(2n)!}{n!n!}\cdot 1 = \frac{(n^2)!}{(n!)^n}.$$

Thus, for any positive integer n, the fraction $\frac{(n^2)}{(n!)^n}$ is the count of how many ways assignments can be made, so it is an integer. Letting $n = 2016$ shows that $(2016!)^{2016}$ must be a divisor of $(2016^2)!$, and solves the first part of the problem.

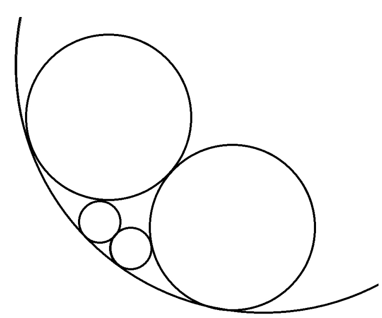
Now imagine that we are assigning the n^2 people into n groups of size n where the team number of each group is not important. Then for every way there is of assigning people to teams numbered $1, 2, 3, \ldots, n$, there are $n!$ equivalent ways of assigning the same people to the same groups but rearranging the order of the teams. Thus, $\frac{(n^2)!}{(n!)^n}$ must be divisible by $n!$ showing that $\frac{(2016^2)!}{(2016!)^{2016}}$ must be divisible by $2016!$ and $\frac{(2016^2)!}{(2016!)^{2017}}$ must be an integer as well.

Note that if $a_1, a_2, a_3, \ldots, a_k$ are nonnegative integers with a sum of b then

$$\binom{b}{a_1\ a_2\ a_3 \ldots a_k} = \binom{b}{a_1}\cdot\binom{b-a_1}{a_2}\cdot\binom{b-a_1-a_2}{a_3}\cdots\binom{a_k}{a_k} = \frac{b!}{a_1!\cdot a_2!\cdot a_3!\cdots a_k!}$$

is called the *multinomial coefficient* and counts the number of ways of placing b people onto k teams of sizes $a_1, a_2, a_3, \ldots, a_k$.

5. Two circles with radius 1 are tangent to each other, two circles with radius 4 are tangent to each other, and each circle with radius 1 is tangent to one of the circles with radius 4. All four of these circles are internally tangent to a larger circle as shown. Show that we can draw a small circle externally tangent to all four of the circles with radii 1 and 4. Find the radius of this small circle.

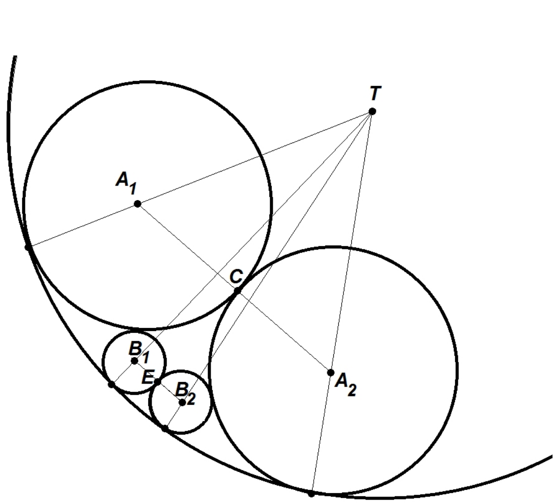


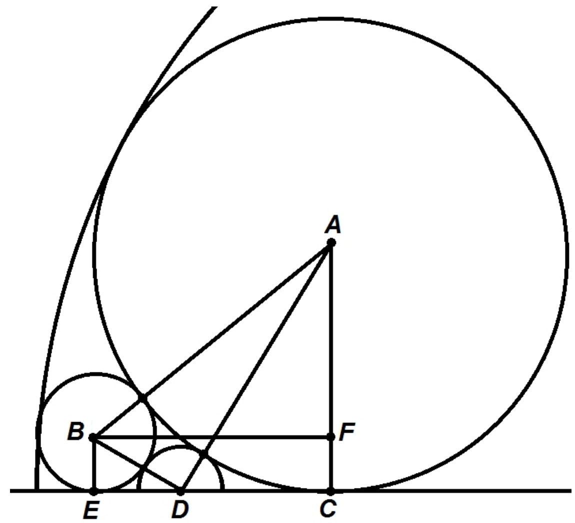
SOLUTION. Denote the center of the large circle by T, the centers of the circles with radius 4 by A_1, A_2 and the centers of the circles with radius 1 by B_1, B_2. Let the two circles with radius 4 be tangent at point C and the two circles with radius 1 be tangent at point E.

Because $A_1T = A_2T$ (both difference of radii), $B_1T = B_2T$ (both difference of radii), and $A_1B_1 = A_2B_2$ (both sum of radii), it follows that $\triangle TA_1B_1$ is congruent to $\triangle TA_2B_2$. Thus,

$\angle A_1TB_1 = \angle A_2TB_2$. Because C bisects $\overline{A_1A_2}$, $\angle A_1TC = \angle A_2TC$ from which $\overline{TC}$ bisects $\angle B_1TB_2$. But $\overline{TE}$ also bisects $\angle B_1TB_2$, so line TE is the same as line TC.

This means that the diameter of the large circle passing through C and E spits the entire figure into two congruent halves. One of these halves is shown in the diagram with the center of the circle with radius 4 marked A, and the center of the circle with radius 1 marked B. Point F is on $\overline{AC}$ so that $\overline{BF} \perp \overline{AC}$.





Because $AC = 4$ and $CF = BE = 1$, $AF = 4 - 1 = 3$. Then by the Pythagorean Theorem, $BF^2 = AB^2 - AF^2 = (4+1)^2 - 3^2 = 16$, so $BF = 4$. Then $AE = \sqrt{4^2 + 4^2} = 4\sqrt{2}$ so that $AE - BE = 4\sqrt{2} - 1 > 4$, and $BC = \sqrt{4^2 + 1^2} = \sqrt{17}$ so that $AC - BC = 4 - \sqrt{17} < 0$. It follows that there is some point D on $\overline{CE}$ so that $AD - BD = 3$. Then if $r = BD - 1 = AD - 4$, a circle centered at D with radius r will be tangent to the circles centered at A and B. By symmetry this circle is tangent to both circles with radius 1 and both circles with radius 4 in the original picture, and is the circle whose radius r we seek.

Applying the Pythagorean Theorem twice, we get that $4 = CE = CD + DE = \sqrt{AD^2 - AC^2} + \sqrt{BD^2 - BE^2} = \sqrt{(4+r)^2 - 4^2} + \sqrt{(1+r)^2 - 1^2}$ which simplifies to $4 = \sqrt{r^2 + 8r} + \sqrt{r^2 + 2r}$. Rearranging and squaring the equation we get

$$r^2 + 2r = (4 - \sqrt{r^2 + 8r})^2 = 16 + r^2 + 8r - 8\sqrt{r^2 + 8r}.$$

Rearranging and squaring again gives

$$64(r^2 + 8r) = (6r + 16)^2 = 36r^2 + 192r + 256,$$

which leads to the quadratic equation

$$7r^2 + 80r - 64 = 0.$$

This equation has one positive solution $r = \frac{-40 + 32\sqrt{2}}{7}$ which is approximately 0.751.