WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET IV (2015-2016)

1. Alice cuts a 100 yard long piece of string into two pieces and holds onto the two cut ends. Then Becky chooses one of the two pieces of string and cuts that piece into two pieces and holds onto the two cut ends. Each player is awarded a prize proportional to the shorter of the two pieces of string she is holding. If both girls are trying to obtain the biggest prize possible, where on the string should Alice make the first cut?

SOLUTION. When Becky cuts a piece of rope in two, she will get a prize proportional to the length of the smaller piece of the two. Thus, she should always cut the piece exactly in half, and she should choose the larger of the two available pieces. (If the available pieces are of the same size then it does not matter which one she picks.)

Suppose Alice cuts the string into pieces of lengths x and $100 - x$ yards where $x \leq 50$ (so that $100 - x \geq x$). Then Becky maximizes her prize by cutting the $100 - x$ long piece of string at its center leaving Alice holding a piece of length x and a piece of length $\frac{100-x}{2}$. So Alice's prize is proportional to $\min(x, \frac{100-x}{2})$.

We want to find $0 \leq x \leq 50$ for which $\min(x, \frac{100-x}{2})$ is the largest possible. This will happen exactly where $x = \frac{100-x}{2}$ or $x = \frac{100}{3}$. Indeed, in that case $\min(x, \frac{100-x}{2}) = \frac{100}{3}$. But if $x < \frac{100}{3}$, then $\min(x, \frac{100-x}{2}) \leq x < \frac{100}{3}$, and if $x > \frac{100}{3}$, then $\min(x, \frac{100-x}{2}) \leq \frac{100-x}{2} < \frac{100}{3}$. Thus, Alice should place her cut a distance of $\frac{100}{3}$ yards from one end of the string.

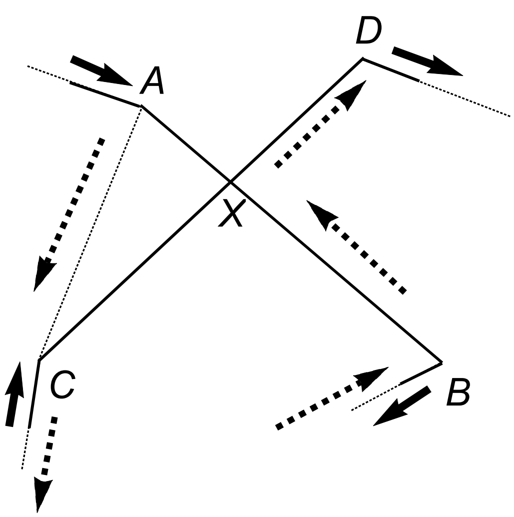
2. There are 8 distinct locations in a field that a runner would like to visit. She has found a path that visits each location exactly once, such that no shorter path also does this. Show that her path does not cross itself. (Assume the locations are points in the plane and any path between them is possible.)

SOLUTION. If the path does not start at one of the locations, then we can shorten the path by starting at the first visited location. Thus the path must start at one of the eight locations, and similarly we can show that it must end at one of the eight locations.

Suppose, for the sake of contradiction, that the path did cross itself. That is, we suppose that there is some point X which the runner visits twice. By our assumption X cannot be any of the eight locations, so there is a location she visits before and after each time she is at X. So let A be the location she visits immediately before she is at point X the first time, and B be the location she visits immediately afterwards. Let C and D be defined analogously for the second time she passes through X.

Since we are considering the shortest possible path, and the shortest path between two points is in a line, her path must include the line segment from A to B, which contains X, and the line segment from C to D, which contains X. Note that X is different from A, B, C and D.

Moreover, the line segments AB and CD cannot be on the same line, as in this case at least one of the four locations A, B, C and D would be visited more than once.



Consider changing her route now as follows. She goes along her original route until she reaches A, then she runs straight from A to C, then she runs backwards along her original route until she reaches B, then runs straight from B to X and then straight to D (using her original path), then continues along her original route. (Have a look at the picture.) Since there are no locations visited between A and X or between C and X, she still visits all 8 locations. We see that her route got shorter by the removal of length $AX + CX$, and longer by AC. However, unless X is on the segment AC, by the triangle inequality we have $AX + XC > AC$, and we have just described a shorter path, which is a contradiction.

Finally, if X is on the segment AC, then since A, X, and B are one the same line, then the line segments AB and CD would be on the same line as well. We have already seen that this is not possible, thus we conclude that the runner's path did not cross itself.

3. Find the smallest positive integer n so that $3n$ is a perfect cube and $5n$ is a perfect 5th power.

SOLUTION. We have to find the smallest positive integer n so that there exist positive integer solutions to the equation system

$$3n = a^3, \qquad 5n = b^5.$$

The first equation shows that a is divisible by 3. This implies that $a^3 = 3n$ is divisible by 3^3 and, thus, n is divisible by 3. The same way we can show that n is divisible by 5. Thus, n can be written as $n = 3^p 5^q r$, where p and q are positive integers, and r is a positive integer not divisible by 3 or 5. This yields the equations

$$3^{p+1} 5^q r = a^3, \qquad 3^p 5^{q+1} r = b^5.$$

If $3^{p+1} 5^q r$ is a perfect cube then $p + 1$ and q must be divisible by 3 and r must be a perfect cube as well. This gives

$$p + 1 = 3p_1, \qquad q = 3q_1,$$

where p_1 and q_1 are positive integers. If $3^p 5^{q+1} r$, then p and $q+1$ must be divisible by 5, and, thus, there must be positive integers p_2 and q_2 with

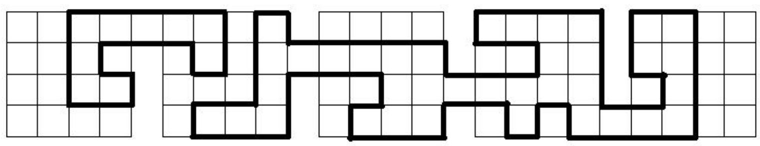
$$p = 3p_1 - 1 = 5p_2, \qquad q + 1 = 3q_1 + 1 = 5q_2.$$

Checking the values $1, 2, 3, \ldots$ we see that the smallest positive integer p_1 for which $3p_1 - 1$ is divisible by 5 is $p_1 = 2$ while the smallest positive integer q_1 for which $3q_1 + 1$ is divisible by 5 is $q_1 = 3$. This means that $p \geq 3 \cdot 2 - 1 = 5$ and $q \geq 3 \cdot 3 = 9$. Thus, $n = 3^p 5^q r$ is at least $3^5 5^9$. But $n = 3^5 5^9$ is a solution:

$$3n = 3^6 \cdot 5^9 = (3^2 \cdot 5^3)^3, \qquad 5n = 3^5 \cdot 5^{10} = (3 \cdot 5^2)^5$$

which means that this is the smallest possible solution.

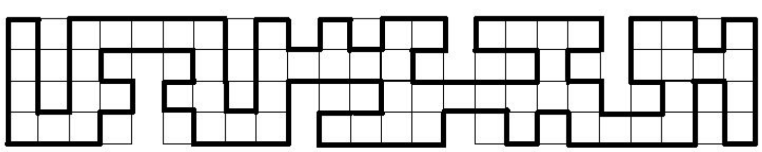
4. The diagram below shows a grid made up of 84 unit squares. A simple closed path on this grid follows the grid lines and forms a closed loop that never passes through any point more than once. Shown with bold are two simple closed paths, one with length 20 and one with length 66. Find the greatest possible length of a simple closed path on this grid.



SOLUTION. The longest such path has length 120. The diagram below shows one possible simple closed path with this length. Now we just have to show that we cannot construct a longer one.

The grid is made up of five 4×4 grids of squares. Each 4×4 grid of squares is a 5×5 grid of corners connected by unit length edges. Any simple closed path that stays within one of these five grids can connect at most 24 of the 25 corners. This can be seen because every path that returns to its starting position must make as many steps to the left as steps to the right, and as many steps up as steps down. Thus, every path must have even length and must visit an even number of corners.

Note also that if a path passes between two 5×5 grids and the path within each grid is longer than one step, then the two edges of the path passing horizontally between the grids can be removed and replaced by the two vertical edges of the square containing the connecting edges. This will break up the closed paths into smaller closed paths which are all fully in their respective 5×5 grids. Since each such smaller closed path can have at most 24 steps, this shows that the complete path had at most $5 \cdot 24 = 120$ steps.



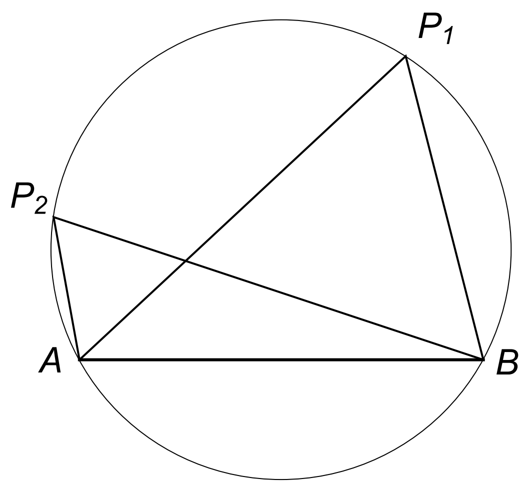
5. Let $A_1A_2A_3A_4A_5A_6A_7$ be a regular heptagon. The diagonals $\overline{A_1A_3}$ and $\overline{A_2A_7}$ intersect at point B, $\overline{A_3A_5}$ and $\overline{A_4A_6}$ intersect at point C, and $\overline{A_1A_4}$ and $\overline{A_3A_7}$ intersect at point D. Show that B, C and D are on the same line. (You should start by sketching a picture.)

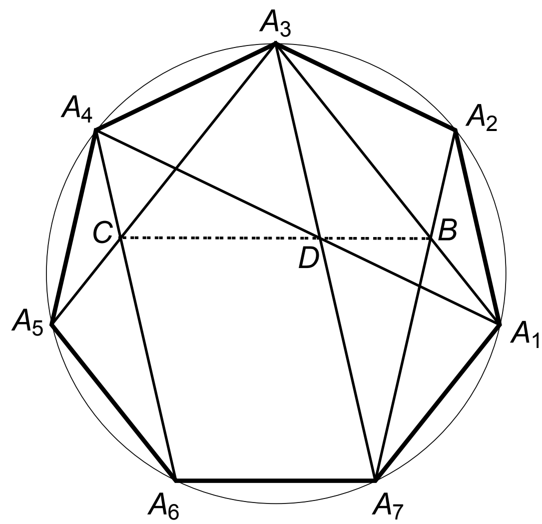
SOLUTION.

We will use the following consequences of the inscribed angle theorem from geometry:

(a) Suppose that $\overline{AB}$ is a chord of a circle, and P_1 and P_2 are points on the circle that are on the same side of $\overline{AB}$. (i.e. on the same arc.) Then the angles $\angle AP_1B$ and $\angle AP_2B$ are equal.

(b) Suppose that P_1 and P_2 are on the same side of the line AB and $\angle AP_1B = \angle AP_2B$. Then A, B, P_1 and P_2 are on the same circle, i.e. ABP_1P_2 forms a cyclic quadrilateral. (This is basically the converse of the previous statement.)





In order to show that B, C and D are on the same line it is enough to show that $\angle BDA_1 = \angle CDA_4$. Consider the regular heptagon $A_1A_2A_3A_4A_5A_6A_7$ with its circumscribed circle.

First observe that $\angle BA_7D = \angle A_2A_7A_3$ and $\angle BA_1D = \angle A_3A_1A_4$ are equal, because they are both inscribed angles corresponding to a side of the regular heptagon. In the quadrilateral BDA_7A_1 we have $\angle BA_7D = \angle BA_1D$ which shows that this is a cyclic quadrilateral. Then the angles $\angle BDA_1$ and $\angle BA_7A_1 = \angle A_2A_7A_1$ must be the same. This means that $\angle BDA_1 = \angle A_2A_7A_1$, which is the inscribed angle corresponding to a side of the regular heptagon.

Next we can check that $\angle CA_4D = \angle A_6A_4A_1$ is the same as $\angle CA_3D = \angle A_5A_3A_7$, as both are inscribed angles corresponding to a diagonal of the regular heptagon of the same size ($A_5A_7 = A_6A_1$). This means that the quadrilateral CDA_3A_4 is cyclic, which in turn implies that $\angle CDA_4 = \angle CA_3A_4$. But $\angle CA_3A_4 = \angle A_5A_3A_4$ which is the same as $\angle A_2A_7A_1$ (either by observing that the triangles $A_5A_3A_4$ and $A_2A_7A_1$ are congruent, or using the inscribed angle theorem again).

Thus, we have shown that $\angle BDA_1 = \angle A_2A_7A_1 = \angle A_5A_3A_4 = \angle CDA_4$ which implies $\angle BDA_1 = \angle CDA_4$ which shows that B, D and C are on the same line.