WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET II (2015-2016)

1. Find every positive integer that is divisible by 8 where the sum of its digits is 7 and the product of its digits is 6.

SOLUTION. The integer clearly must have at least two digits. If we have to write 6 as the product of at least two positive integers then we can do that as $6 \cdot 1 \cdot (1 \cdots 1)$ (where we need at least one factor of 1) or as $3 \cdot 2 \cdot (1 \cdots 1)$ (where we may or may not have some factors of 1). Since the sum of digits must be 7, we either have two digits: 6 and 1, or four digits: 3, 2, 1, 1. In the first case there are two possible numbers with these digits: 16 and 61, and only the first one is divisible by 8. In the second case there are several choices, but only those ending with 2 are even and have a chance to be divisible by 8. These are 1132, 1312, 3112. Out of these only 1312 and 3112 are divisible by 8. Thus there are three such integers: $16, 1312$ and 3112.

2. We choose two different numbers randomly out of the numbers $1, 2, \ldots, 50$. We assume that each possible pair of numbers is equally likely. What is the probability that the product of the two chosen numbers is divisible by 10?

SOLUTION. We will consider the pairs (a, b) where both a and b are between 1 and 50, and $a \neq b$. (Note that we count $(1, 2)$ and $(2, 1)$ as two different pairs.) There are $50 \cdot 49$ such pairs since for every choice of a there are 49 other choices for b.

We will now count those pairs (a, b) where ab is divisible by 10. There are several ways to do this, but we must make sure that we count every such pair exactly once. We will look at a couple of cases based on the value of a:

- If a is divisible by 10 then ab is also divisible by 10. There are 5 multiples of 10 up to 50. That's 5 choices for a, and since b can be anything (apart from a), there are $5 \cdot 49$ such pairs.
- If a is divisible by 5, but not by 10, then b must be an even number. There are 5 choices for a (as there are 10 multiples of 5, and only 5 of them are not divisible by 10). For each choice of a there are 25 choices for b (that's the number of even numbers up to 50). Note that a and b cannot be equal since a is not even, and b is even. Thus we have $5 \cdot 25$ such pairs.
- If a is divisible by 2 but not by 10 (i.e. not by 5), then b must be a multiple of 5. There are 20 choices for a (there are 25 even numbers, 5 of them are also divisible by 5). For every choice of a there are 10 choices for b since there are 10 multiples of 5 up to 50 (and none of them can be equal to a, which is not divisible by 5). This gives $20 \cdot 10$ such pairs.
- If a is divisible neither by 2 nor by 5, then b must be divisible by 10. There are 20 numbers up to 50 that are not divisible by 2 or 5 (there are 25 odd numbers, 5 of them are divisible by 5). That gives 20 choices for a, and we have 5 choices for b (the multiples of 10) which gives $20 \cdot 5$ such pairs.

We have looked at all possibilities, and the total number of pairs with ab divisible by 10 is $5 \cdot 49 + 5 \cdot 25 + 20 \cdot 10 + 20 \cdot 5 = 670$. This means that the probability of choosing two numbers between 1 and 50 so that their product is a multiple of 10 is $\frac{670}{50 \cdot 49} = \frac{67}{245}$.

Another possible solution: The product ab is divisible by 10 if one of the following two cases holds:

- one of the two numbers is divisible by 10,
- none of the numbers is divisible by 10, one of them is even, one of them is divisible by 5

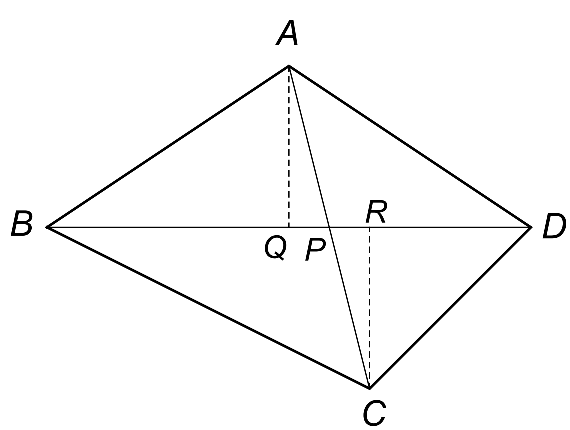
The number of pairs of numbers corresponding to the first case are $5 \cdot 49 + 49 \cdot 5 - 5 \cdot 4 = 470$ (counting whether the first or the second number is a multiple of 10, and then subtracting the cases when both of them are that have been counted twice). The number of pairs of numbers corresponding to the second case is $(25-5) \cdot (10-5) \cdot 2 = 200$. (There are $25 - 5 = 20$ even numbers up to 50 which are not divisible by 5, and $10-5 = 5$ multiples of 5 which are not divisible by 2, and the even number can be either the first or the second of the pair.) This gives a total number of 670 pairs (again) where ab is divisible by 10.

3. The real numbers a, b, c, and d are each less than or equal to 12. The polynomial $p(x) = ax^3 + bx^2 + cx + d$ satisfies $p(2) = 2$, $p(4) = 4$, and $p(6) = 6$. What is the maximum possible value of $p(10)$?

SOLUTION. Since $p(x)-x$ is a cubic polynomial that is 0 for $x = 2$, $x = 4$ and $x = 6$, there must be a real number k such that $p(x)-x = k(x-2)(x-4)(x-6)$ or $p(x) = k(x^3-12x^2+44x-48)+x$. Then $a = k$, $b = -12k$, $c = 44k + 1$, and $d = -48k$. In order for $p(10) = k(10-2)(10-4)(10-6) + 10 = 192k + 10$ to be maximal, we should choose k as large as possible. We know that the numbers $k, -12k, 44k+1, -48k$ are all at most 12. From $44k+1 \leq 12$ we get $k \leq \frac{1}{4}$. Moreover, for $k = \frac{1}{4}$ the other three numbers $k, -12k, -48k$ are all at most 12. Thus the largest possible value of k is $\frac{1}{4}$, and the largest possible value of $p(10)$ is $\frac{1}{4}(10-2)(10-4)(10-6) + 10 = 58$.

4. Each of the diagonals of a convex quadrilateral $ABCD$ divide the quadrilateral into two equal area triangles. Show that $ABCD$ is a parallelogram.

SOLUTION. We will show that the diagonals $\overline{AC}$ and $\overline{BD}$ bisect each other. From this it follows that $ABCD$ is a parallelogram. Let P be the intersection point of the diagonals. Denote the foot of the altitude dropped from A to BD in the triangle $\triangle ABD$ by Q. Denote the foot of the altitude dropped from C to BD in the triangle $\triangle CDB$ by R.



The areas of $\triangle ABD$ and $\triangle CDB$ are the same. Moreover, these triangles have a common side in $\overline{BD}$. Thus the corresponding altitudes, $\overline{AQ}$ and $\overline{CR}$, must have the same length. But this means that the right triangles $\triangle AQP$ and $\triangle CRP$ are congruent to each other (the angles match up, and $\overline{AQ} = \overline{CR}$). This leads to $\overline{AP} = \overline{PC}$. The same way we can show that $\overline{BP} = \overline{PD}$ which shows that the diagonals of $ABCD$ bisect each other. Thus triangles $\triangle APB$ and $\triangle CPD$ are congruent, from which it follows that $\angle BAP = \angle DCP$. From this it follows that $\overline{AB}$ and $\overline{CD}$ are parallel. An analogous argument shows $\overline{AD}$ and $\overline{BC}$ are parallel.

5. A collection of line segments contained in a square of side length one inch is said to be *shading* if every straight line which intersects the square also intersects one of the line segments. Think of the line segments as painted walls in a house otherwise made of glass. For a shading set, a flashlight beam cannot pass through the house. E.g. if we take three of the sides of the square, or its two diagonals, then we have shading sets. If we take only two of its sides then we get a set which is not shading. A shading set need not consist of connected line segments. The length of the shading set in our two examples is 3 and $2\sqrt{2} \approx 2.828$ inches. Find a shading set which has length less than 2.75 inches. You get an extra point if you can find one with total length less than 2.71 inches.

SOLUTION. We give two examples, one with total length $1 + \sqrt{3} \approx 2.732$ inches, and another with total length $2 + \sqrt{2}/2 \approx 2.707$ inches.

For the first example we take the midpoints E and F of the sides $\overline{BC}$ and $\overline{AD}$. We then consider the points P and Q on $\overline{EF}$ for which $\overline{EP} = \overline{FQ} = \frac{1}{2\sqrt{3}}$. The Pythagoraen Theorem then gives $\overline{BP} = \overline{CP} = \overline{AQ} = \overline{DQ} = \frac{1}{\sqrt{3}}$. Our shading set is the collection of line segments $\overline{BP}, \overline{CP}, \overline{AQ}, \overline{DQ}, \overline{PQ}$. It is easy to check that the total length is $4 \cdot \frac{1}{\sqrt{3}} + (1 - \frac{1}{\sqrt{3}}) = 1 + \sqrt{3}$. Since our set connects all four vertices of the square, any line intersecting the square will also intersect the set.

For the second example we consider the sides $\overline{AB}$ and $\overline{BC}$, and a segment from the opposite corner D of the square to the center O of the square. Then $AB + BC + DO = 2 + \sqrt{2}/2 \approx 2.707$. If a line intersects $\overline{AB}$ or $\overline{BC}$, then clearly it intersects one of our chosen segments. Otherwise, if a line ℓ intersects the square it must intersect $\overline{CD}$ and $\overline{DA}$. Then to get from triangle CDO to triangle DAO is must intersect segment $\overline{DO}$.