## WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH
## SOLUTIONS TO PROBLEM SET I (2015-2016)

**1.** Show that out of seven integers you can always find four so that the sum of these numbers is divisible by 4.

**SOLUTION.** We start by noting that out of any three integers we can always find two where the sum is even. This is because out of the three numbers there will be at least two even or at least two odd and in both of these cases the sum of these two numbers is even.

Now consider the seven integers. Take (any) three of them; by our previous observation we can choose two out of these three where the sum is even. We pair these two numbers up and set them aside. This leaves 5 numbers. Again we take (any) three, choose two of them that add up to an even number, pair them up, and set them aside. Out of the remaining three integers we can again produce a pair where the sum is even. This gives us three pairs of numbers so that the sum is even in each pair. Now consider the three sums. These are even numbers, so dividing them by two produces three integers. But out of these three integers we can again find two where the sum is even. If we now add the numbers in the corresponding pairs (which gives four numbers), the sum will be twice an even number, which is a multiple of 4.

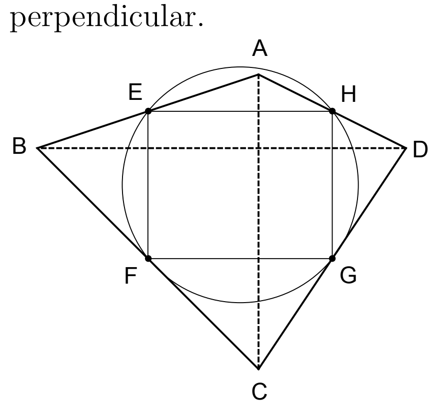
It is not hard to produce an example for six integers where it is impossible to find four with sum divisible by 4. For example, $0, 0, 0, 1, 1, 1$ works.

**2.** There is a circle that passes through the midpoints of all four sides of the quadrilateral $ABCD$. Show that the diagonals $\overline{AC}$ and $\overline{BD}$ are perpendicular.

**SOLUTION.** Denote the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$ by $E, F, G$ and $H$, respectively. For any triangle the line connecting the midpoints of two sides is parallel to the third side (this can be checked using similar triangles). Thus $\overline{EF}$ is parallel to $\overline{AC}$ (because of the triangle $ABC$) and $\overline{GH}$ is parallel to $\overline{AC}$ (because of the triangle $ACD$). This shows that $\overline{EF}$ is parallel to $\overline{GH}$. Similarly, $\overline{FG}$ is parallel to $\overline{HE}$ (because both are parallel to $\overline{BD}$).

This shows that the opposing sides of the quadrilateral $EFGH$ are parallel to each other. Hence it must be a parallelogram. This quadrilateral is also cyclic (by the conditions of the problem), but the only parallelograms that are cyclic are rectangles. This is because the opposite angles of a cyclic quadrilateral must add up to 180°, while the opposite angles of a parallelogram must be equal.

Therefore $EFGH$ is a rectangle. Since $\overline{AC}$ and $\overline{BD}$ are parallel to $\overline{EF}$ and $\overline{FG}$, respectively, this shows that $\overline{AC}$ and $\overline{BD}$ are perpendicular.



**3.** How many three-digit positive integers satisfy the condition that any two consecutive digits differ by exactly one?

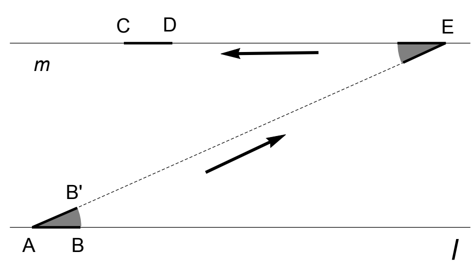
**SOLUTION.** If the first digit of such a number is $a$ then the three digits (in order) are either $(a, a+1, a+2)$, $(a, a+1, a)$, $(a, a-1, a-2)$ or $(a, a-1, a)$. For each one of these possibilities we have to count how many ways we can choose $a$ between 1 and 9 so that the second and third digits are between 0 and 9. It is easy to check that the possible values for $a$ in the fours cases are

$$\begin{array}{llll} (a, a+1, a+2): & 1 \le a \le 7, & (a, a+1, a): & 1 \le a \le 8 \\ (a, a-1, a-2): & 2 \le a \le 9, & (a, a-1, a): & 1 \le a \le 9. \end{array}$$

Altogether this gives $7 + 8 + 8 + 9 = 32$ possibilities.

**4.** Let $\ell$ and $m$ be two parallel lines one inch apart. We choose the line segment $\overline{AB}$ on $\ell$ and the line segment $\overline{CD}$ on $m$, each 2 inches long. Then for any position of the line segments we can move $\overline{AB}$ so as to coincide with $\overline{CD}$ in a way that the moving segment sweeps out a region of area exactly 2 square inches as it moves. One way to do this is by moving $\overline{AB}$ parallel to itself within the parallelogram $ABCD$. Prove that if $\ell$ and $m$ were 100 miles apart (instead of 1 inch), it would still be possible to move $\overline{AB}$ to coincide with $\overline{CD}$ while sweeping out a region of no more than 2 square inches.

**SOLUTION.** We describe one possible way of moving $\overline{AB}$ to $\overline{CD}$. We first rotate $\overline{AB}$ around $A$ just slightly, into the segment $\overline{AB'}$ pointing towards $m$. Denote the intersection of the line $AB'$ with the line $m$ by $E$. Then we slide the segment $\overline{AB'}$ along the line $AB'$ towards $m$, until it hits $m$ at $E$. Then we rotate the segment around $E$ again (in the opposite direction as before) until the segment lies on $m$. Then we translate the segment along $m$ until it coincides with $\overline{CD}$. When we translate the segment along a line, since the line has 0 area, no further area is swept out. By making the angle of rotation smaller, we can make the area swept out during the rotation as small as a percentage of a circle of radius 2 as we desire.




**5.** Show that there is a finite list of distinct positive integers, each containing the digit 1, such that the sum of their reciprocals is more than 100.

**SOLUTION.** Consider all $k$ digit numbers starting with the digit 1 (with $k \ge 1$). There are $10^{k-1}$ such numbers, as each digit starting from the second and ending at the $k^{th}$ can be chosen as any of the 10 possible digits. If a $k$ digit number starts with 1, then it is less than $2 \times 10^{k-1}$, so the sum of reciprocals of such numbers is at least $10^{k-1} \times \frac{1}{2 \times 10^{k-1}} = \frac{1}{2}$.

If we now collect all such numbers for all $k$ between 1 and 200, then the total sum of reciprocals will be more than $200 \cdot 1/2 = 100$.