WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET V (1999-2000)

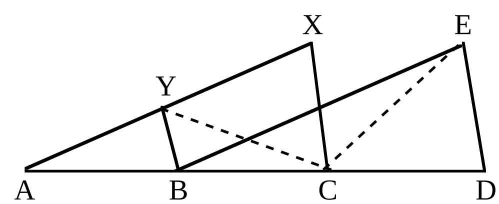
1. Find all positive integers n so that $\frac{1}{n}$ is the repeating decimal

$$\frac{1}{n} = .abcabcabc\cdots = .\overline{abc}$$

with a, b and c distinct digits between 0 and 9.

SOLUTION. If $\frac{1}{n} = .\overline{abc}$, then $\frac{1000}{n} = abc.\overline{abc}$ and thus $\frac{999}{n} = \frac{1000}{n} - \frac{1}{n} = abc$. In other words, we have $n(abc) = 999$, where abc is a three digit integer with distinct digits. Now $999 = 3^3 \cdot 37$, so the factors of 999 are 1, 3, 9, 27, 37, 111, 333 and 999. The only factors with distinct digits are therefore $27 = 027$ and $37 = 037$, so it follows that $n = \frac{999}{abc} = \frac{999}{27} = 37$ or $n = \frac{999}{37} = 27$.

2. In the figure, $\overline{ABCD}$ is a straight line with $AB = BC = CD = 2$. Also $FA = DE = 2$, $BE = 4$, and $FC = CE$. Compute the distance FB.



SOLUTION. Choose point X as shown with $AX = 4$ and $XC = 2$. Let Y be the midpoint of $\overline{AX}$ so that $YA = 2$. Since B is the midpoint of $\overline{AC}$, it follows that $\triangle YAB$ is similar to $\triangle XAC$ with ratio 1 : 2. Hence $YB/XC = 1/2$ and $YB = XC/2 = 1$. We show now that $Y = F$.

By assumption, $BE = AC = 4$, $BD = AX = 4$ and $XC = DE = 2$. Thus by the SSS congruence criterion, $\triangle XAC$ is congruent to $\triangle DBE$. Since $\overline{CY}$ is the median to side $\overline{AX}$ in the first triangle, and $\overline{EC}$ is the median to side $\overline{BD}$ in the second, it follows that $CY = EC$. Thus Y is the unique point above the line $\overline{AD}$ where the circle of radius 2 centered at A and the circle of radius CE centered at C intersect. But F has this property, so $F = Y$ and $FB = YB = 1$.

3. Consider the sequence of integers $x_1 = 34,\ x_2 = 334,\ x_3 = 3334,\ \ldots,\ x_n = 33\cdots334,\ \ldots$ where the first n digits of x_n are 3s and the units digit is a 4. Compute the number of digits that are equal to 3 in the number $9(x_n)^3$.

SOLUTION. Since $x_n - 1 = 33\cdots333$, we have $x_n - 1 = (10^{n+1} - 1)/3$ and hence $x_n = (10^{n+1} + 2)/3$. Thus

$$\begin{aligned} 9(x_n)^3 &= 9[(10^{n+1} + 2)/3]^3 = (1/3)(10^{n+1} + 2)^3 \\ &= (1/3)(10^{3(n+1)} + 6\cdot 10^{2(n+1)} + 12\cdot 10^{n+1} + 8) \\ &= (10^{3(n+1)} - 1)/3 + 2\cdot 10^{2(n+1)} + 4\cdot 10^{n+1} + 3. \end{aligned}$$

Note that $(10^{3(n+1)} - 1)/3$ has all $3(n+1)$ digits equal to 3 and that the remaining three summands above add 2, 4, and 3 to the $10^{2(n+1)}$, $10^{(n+1)}$ and units digit, making them 5, 7 and 6, respectively. In other words,

$$9(x_n)^3 = \underbrace{33\cdots3}_{n}5\underbrace{33\cdots3}_{n}7\underbrace{33\cdots3}_{n}6$$

and there are precisely $3n$ digits equal to 3.

4. Do there exist infinitely many triples (x, y, z) of real numbers which satisfy the system of equations

$$x^2 + xy + y^2 = y^2 + yz + z^2 = z^2 + zx + x^2 = 3$$

Justify your answer.

SOLUTION. There are infinitely many solutions. To see this, first note that

$$(y^2 + yz + z^2) - (x^2 + xy + y^2) = (z - x)(x + y + z)$$
$$(z^2 + zx + x^2) - (x^2 + xy + y^2) = (z - y)(x + y + z)$$

In particular, if $x^2 + xy + y^2 = 3$ and if $x + y + z = 0$, then the triple (x, y, z) will satisfy $x^2 + xy + y^2 = y^2 + yz + z^2 = z^2 + zx + x^2 = 3$. Now if y is given, then we can solve the quadratic equation $x^2 + xy + y^2 = 3$ for x using the quadratic formula to obtain

$$x = \frac{-y \pm \sqrt{3(4 - y^2)}}{2}$$

Thus, for any real number y with $-2 \leq y \leq 2$, we obtain at least one real number x satisfying $x^2 + xy + y^2 = 3$. Furthermore, if we set $z = -(x + y)$, then $x + y + z = 0$, and (x, y, z) is a solution to the original system of equations.

5. Suppose

$$1 = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_m}$$

where $a_1, a_2, \ldots, a_m$ are distinct positive integers. If the largest of the a_is is equal to $2p$ for some prime p, find the set $\{a_1, a_2, \ldots, a_m\}$.

SOLUTION. For convenience, let $a_1 = 2p$. Then

$$\frac{2p-1}{2p} = 1 - \frac{1}{2p} = 1 - \frac{1}{a_1} = \frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_m} = \frac{b}{a_2 a_3 \cdots a_m}$$

for some integer b. Thus $2pb = (2p - 1)a_2 a_3 \cdots a_m$, and p divides $(2p - 1)a_2 a_3 \cdots a_m$. But p does not divide $2p - 1$, so the prime p must divide some a_j with $j \geq 2$. Say p divides a_2. Now $a_1 = 2p$ is the largest of the a_is and the only positive multiple of p less than $2p$ is p itself. Thus $a_2 = p$ and p cannot divide any of the remaining $a_3, a_4, \ldots a_m$. Finally

$$\frac{2p-3}{2p} = 1 - \frac{1}{2p} - \frac{1}{p} = 1 - \frac{1}{a_1} - \frac{1}{a_2} = \frac{1}{a_3} + \frac{1}{a_4} + \cdots + \frac{1}{a_m} = \frac{c}{a_3 a_4 \cdots a_m}$$

for some integer c, so $2pc = (2p - 3)a_3 a_4 \cdots a_m$ and p divides the product $(2p - 3)a_3 a_4 \cdots a_m$. But the prime p does not divide $a_3, a_4, \ldots, a_m$, so p must divide $2p - 3$, and hence $p = 3$. It follows that $a_1 = 6$, $a_2 = 3$ and the remaining a_is, being less than 6, belong to the set $\{1, 2, 4, 5\}$. Since the reciprocals of these a_is must sum to $1 - \frac{1}{6} - \frac{1}{3} = \frac{1}{2}$, we see that no $a_i = 1$. Thus, since $\frac{1}{4} + \frac{1}{5} < \frac{1}{2}$, the only possibility is $a_3 = 2$ and $\{a_1, a_2, \ldots, a_m\} = \{6, 3, 2\}$.